

# Mathematical Case Studies:

## Basic Analysis\*

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### Abstract

This **ProofPower-HOL** script contains definitions and proofs concerning some of the basics of analysis. The material covered includes: polynomial functions on the real numbers, limits of sequences, continuity of functions of real variables, differentiation, limits of function values, uniform convergence of limits of functions, series and power series and their use in defining the exponential, logarithm and sine and cosine functions; the other circular trigonometric functions and the hyperbolic trigonometric functions and a selection of the inverse functions; definition and basic properties of the Henstock-Kurzweil gauge integral including the fundamental theorem of the calculus; calculation of the areas of some simple plane sets.

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**Reference: LEMMA1/HOL/WRK066; Current git revision: fadb4d2**

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\*First posted 11 April 2004; for full changes history see: <https://github.com/RobArthan/pp-contrib>.

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## To Do

- Make systematic use of open and closed intervals in the statements of the basic theorems on limits etc. (A start has been made on this, recent results having been stated using intervals).
- $\rightarrow$  Consider adding a more detailed informal discussion of the extension of Mitchell's "axiomatic" approach to the trigonometric functions to give the results on periodicity.
- Consider adding the theory of uniform continuity.
- Add more results on the gauge integral (in particular, the calculational results one needs for integrals like the ones that define the Laplace transform).

## Acknowledgments

My thanks to Robin Chapman, Matthew Frank, Hanne Gottliebse, John Harrison and Roger Jones for their helpful correspondence.

## References

- [1] Hanne Gottliebse. *Automated Theorem Proving for Mathematics: Real Analysis in PVS*. PhD thesis, University of St. Andrews, 2001.
- [2] John Harrison. *Theorem Proving with the Real Numbers*. Technical report, University of Cambridge Computer Laboratory, 1996.
- [3] Douglas S. Kurtz and Charles W. Swartz. *Theories of Integration: the Integrals of Riemann, Lebesgue, Henstock-Kurzweil, and MacShane*. World Scientific, 2004.
- [4] Barry M. Mitchell. *Calculus without Analytic Geometry*. D.C. Heath & Co., 1969.
- [5] Walter Rudin. *Principles of Mathematical Analysis*. McGraw-Hill Book Company, 1974.

# 1 INTRODUCTION

This document defines a theory *analysis* giving a formalisation in ProofPower-HOL of the basics of analysis on the real number line. The subjects covered are the basic facts about:

- polynomial functions on the real numbers
- limits of sequences
- continuity of functions
- differentiation
- limits of function values
- uniform convergence of limits of functions
- series and power series
- special functions: exponential function, natural logarithm, sine and cosine
- L'Hôpital's Rule and some applications to particular limits
- further special functions: square root, circular and hyperbolic trigonometric functions.
- the Henstock-Kurzweil gauge integral and the fundamental theorem of the calculus with an application to the areas of plane sets.

This is a similar enterprise to John Harrison's HOL treatment of the calculus described in chapter 3 of his thesis [2], but was inspired by rather different motivations. This leads to some differences in methods and in the choice of subject matter. In particular, I am interested in geometric and topological applications of the material, particularly the trigonometric functions and not so much in computational aspects or formalised numerical analysis.

On the one hand, the geometrical motivations mean that there is less emphasis on certain results that are important in numerical analysis (e.g., Taylor's theorem). On the other hand, they make it more attractive to prove somewhat more general results, e.g., the result on "differentiating under the limit sign" that leads to the general theorem on the convergence of power series and their derivatives (Harrison proves these results for specific power series by less general methods).

Another difference of approach in comparison with Harrison's is encouraged by the ProofPower tradition of using the HOL constant specification facility to specify mathematical objects implicitly wherever that seems to accord with good mathematical practice. For example, we prefer to specify the transcendental functions using differential equations rather than power series. In the same spirit, Archimedes' constant,  $\pi$ , is characterised as the positive generator of the group of zeroes of the sine function.

The work was begun late in 2001 originally just to exercise the real number theory in ProofPower-HOL on a purely mathematical application. That work had some success with polynomials, limits, continuity and derivatives and so on but foundered due to lack of time and a technical difficulty with the traditional textbook approach to the result on differentiating under the limit sign mentioned above. This technical difficulty arises from the fact that, while, for simplicity, I deal only with two-sided derivatives, the usual textbook proofs involve working with functions defined on a closed interval and make use of the derivatives at the ends of the interval where they are inherently one-sided. In fact, with hindsight, this turns out not to be a problem: essentially the same results can be formulated in equal generality using open intervals and two-sided limits.

Motivated by some more recent work on topology and geometry, I returned to the document late in 2003 and found the missing results not too difficult to supply. This leads immediately to the applications in defining the basic transcendental functions. This latter material constitutes quite a large fraction of the proof scripts (about 15% of the current total), but the material is very straightforward and the development of the properties of these functions from their specifications via differential equations took only a few evenings' work.

Since 2003, the treatment has grown as time and inclination has permitted, often motivated by Freek Wiedijk's list of 100 challenge problems. In particular, the theory of the Henstock-Kurzweil has been developed and applied in various ways, e.g., to calculate the area of a circle.

Undertaking this kind of work is a mathematical activity of an unusual, often entertaining, albeit sometimes frustrating nature. It is rather like preparing the material for a course or a textbook with the assistance of an amanuensis who is an *idiot savant* of an unusual kind. Firstly, he insists on and gets absolute editorial control over what purports to be a proof: every proof step is checked with complete accuracy and no lacunae slip his attention. On the more constructive side, he is capable of amazing feats of calculation: so for example, before you start he can already (almost) infallibly decide the insolubility of a system of linear constraints. Reducing problems to such systems becomes second nature as the work progresses. Similarly, once rules for differentiating sums, product and so on are available, the amanuensis can readily be taught how to apply them to arbitrarily complex expressions. He can then determine, for example, that the derivative of  $\cos(x)^2 + \sin(x)^2$  vanishes without any work on your part at all.

Another remarkable property of the amanuensis is that there is absolutely no need to present ideas in simple-minded ways to help him get used to them. I imagine that many undergraduates encountering the calculus for the first time would be somewhat fazed by the definition of the set of polynomial functions as the smallest set of functions containing constants and the identity function and closed under addition and multiplication. This is the very first definition we give (see section 2.2). I imagine most undergraduates would be completely fazed by immediately having to derive a principle of proof by induction over this set and putting that principle to work to derive useful general theory. Similarly, a lot of geometric notions and even a little topological group theory are needed to motivate the differential equations for the transcendental functions in section 2.15, but the amanuensis does not need this: as soon as you convince him that the definitions are consistent (by showing that the power series expansions satisfy the equations), he will happily help you derive the usual algebraic properties, proving things like the sum rule for sine and cosine without tears.

The excellent textbooks by Rudin [5] and Mitchell [4] are cited several times in the discussion. In particular, the formal treatment of power series is based on Rudin and the "axiomatic" derivation of the properties of the trigonometric functions using the defining differential equations as the axioms is based on Mitchell. I do not know of a reference for the analogous treatment of the exponential function, and (re-)invented this for myself (along with most of the more elementary theorems).

## 2 AN OVERVIEW OF THE THEORY

In this section, we give the formal definitions of the HOL constants with which the theory deals and discuss the theorems that we will prove about them. To keep our use of the ProofPower document preparation system simple, we simply identify the theorems by name and refer the reader to the theory listing in section A on page 39 for the formal statements of the theorems. There is an index to the formal definitions and the theory listing in section 2.18.4 on page 93.

This document is a ProofPower literate script. It contains all the metalanguage (ML) commands required to create a ProofPower-HOL theory (called "analysis"), populate it with the formal defini-

tions and prove and record all the theorems. The proof commands are suppressed from the printed form of the document. ML proof scripts are not particularly informative even to the expert eye, except when they are brought alive by replaying them interactively.

In sections 2.2 to 2.18 below, the various notions with which we are concerned are specified. These sections also discuss the theorems proved. Theorems are referenced by the names under which they are saved in the theory and an index is provided to help you find the theorems in the theory listing.

There are several ways of motivating this material and my preference is for a topological or geometric approach. From that point of view, almost all of sections 2.2 to 2.14.4 can be viewed as leading towards the results on the transcendental functions in section 2.15 below.

Section 2.16 presents l'Hôpital's rule and was motivated by the need to calculate the limits that appear in the theory of the Laplace transform. It is in any case pleasant to have the standard results about the asymptotic behaviour of  $\exp$  and  $\log$ .

Section 2.18 presents the theory of integration following the Henstock-Kurzweil approach. Again this was motivated by a desire to work with transforms (for which the Henstock-Kurzweil gauge integral is particularly well-suited since it gives a uniform definition for integration over both bounded and unbounded intervals). As an application a notion of area for plane sets is defined and the areas of rectangles and circles are calculated.

The material is presented below in a bottom-up order in sections 2.2 to 2.18 below.

## 2.1 Technical Prelude

First of all, we must give the the ML commands to introduce the new theory “analysis” as a child of the theory  $\mathbb{R}$  of real numbers. The parents of the theory are the theory  $\mathbb{R}$  of real numbers and the theory “fin\_set” which introduces the notion of finiteness (used here in the definition of compactness and in proofs about compact sets). We set up a proof context (which defines parameters to most of the proof infrastructure) by merging together: “basic\_hol1” for assistance with the predicate calculus, natural numbers, ordered pairs, lists etc.; “sets\_alg” for sets; and “ $\mathbb{R}$ ” for real numbers. The resulting proof context will automatically prove trivialities such as  $Hd [1/2; 1/3] \in \{x \mid x^2 < \mathbb{N}R 1\}$ , and will generally carry out useful simplifications to most problems expressed using this sort of vocabulary. (Here ‘ $[1/2; 1/3]$ ’ is the ProofPower-HOL syntax for writing the list whose elements are 1/2 and 1/3 and  $Hd$  is the operator that takes the head element of a list, so that the membership assertion reduces to the true inequality  $1/4 < 1$ ).

SML

```
|force_delete_theory"analysis" handle Fail _ => ();
|open_theory"ℝ";
|set_merge_pcs["basic_hol1", "'sets_alg", "'ℝ"];
|new_theory"analysis";
|new_parent"fin_set";
```

## 2.2 Polynomials

We define the polynomial functions on the real numbers to be the smallest set of functions containing the constant functions and the identity function and closed under (point-wise) addition and multiplication.

Here and throughout this document, we use the ProofPower Z-like notation for introducing new HOL constants. In this notation, one gives type ascriptions for the constant or constants being defined

(in this case *PolyFunc* which is a set of real functions of a real variable) and then a predicate giving their defining property (in this case an equation giving the value of the constant). This maps onto a call of the primitive definitional principle *const\_spec*. This principle requires an existence proof for the constants being introduced. The **ProofPower**-HOL infrastructure includes a range of procedures for discharging the existence proofs and these will automatically discharge the proof obligations for most of the definitions in this document

If the existence proof cannot be discharged automatically, the actual defining property used in the call to *const\_spec* is a property that can trivially be proved consistent and that is logically equivalent to the desired defining property if that is consistent. The existence proof can then be conducted under manual control at a later stage. This has been done for all the constants in this document which are not handled automatically. Indeed, the proof of consistency of the definitions of the transcendental functions and  $\pi$  in section 2.15 below can be viewed as the *raison d'être* for the development of the differential calculus and for the theory of power series.

HOL Constant

***PolyFunc*** :  $(\mathbb{R} \rightarrow \mathbb{R})$  SET

---

*PolyFunc* =  $\bigcap$

{

$A$

$(\forall c \bullet (\lambda x \bullet c) \in A)$

$(\lambda x \bullet x) \in A$

$(\forall f \ g \bullet f \in A \wedge g \in A \Rightarrow (\lambda x \bullet f \ x + g \ x) \in A)$

$(\forall f \ g \bullet f \in A \wedge g \in A \Rightarrow (\lambda x \bullet f \ x * g \ x) \in A)$  }

We will show that every polynomial function can be represented as a (point-wise) sum,  $\lambda x \bullet a_0 + a_1x + a_2x^2 + \dots a_nx^n$ , for some list of coefficients  $[a_0; a_1; a_2 \dots; a_n]$ . The following function maps such a list of coefficients into the polynomial function it represents.

HOL Constant

***PolyEval*** :  $\mathbb{R}$  LIST  $\rightarrow$   $(\mathbb{R} \rightarrow \mathbb{R})$

---

$(\forall x \bullet \text{PolyEval } [] \ x = \text{NR } 0)$

$\wedge (\forall c \ l \ x \bullet \text{PolyEval } (\text{Cons } c \ l) \ x = c + x * \text{PolyEval } l \ x)$

We now define the operations on lists of coefficients that correspond to addition of polynomials ...

HOL Constant

***PlusCoeffs*** :  $\mathbb{R}$  LIST  $\rightarrow$   $\mathbb{R}$  LIST  $\rightarrow$   $\mathbb{R}$  LIST

---

$(\forall l \bullet \text{PlusCoeffs } [] \ l = l)$

$\wedge (\forall l \bullet \text{PlusCoeffs } l \ [] = l)$

$\wedge (\forall c1 \ l1 \ c2 \ l2 \bullet$

$\text{PlusCoeffs } (\text{Cons } c1 \ l1) (\text{Cons } c2 \ l2) =$

$\text{Cons } (c1 + c2) (\text{PlusCoeffs } l1 \ l2))$

... and to multiplication of one polynomial by another.



HOL Constant

**TimesCoeffs** :  $\mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST}$

---

$(\forall l \bullet \text{TimesCoeffs } [] \ l = [])$   
 $\wedge (\forall c \ l1 \ l2 \bullet$   
 $\text{TimesCoeffs } (\text{Cons } c \ l1) \ l2 =$   
 $\text{PlusCoeffs } (\text{Cons } (\mathbb{N} \ 0) (\text{TimesCoeffs } l1 \ l2)) (\text{Map } (\lambda x \bullet c * x) \ l2) )$

The following is useful in forming a polynomial whose list of coefficients is given by a function:

SML

`declare_infix(310, "To");`

HOL Constant

**\$To** :  $(\mathbb{N} \rightarrow 'a) \rightarrow \mathbb{N} \rightarrow 'a \text{ LIST}$

---

$(\forall f \bullet f \ \text{To} \ 0 = [])$   
 $\wedge (\forall f \ n \bullet f \ \text{To} \ (n+1) = f \ \text{To} \ n @ [f \ n])$

After one or two technicalities, the main results on polynomials begins with a pattern which will appear often. We are interested in showing that every polynomial function has a certain property, in this case the property of being represented by a list of coefficients. To prove this we investigate how the constant functions, the identity function, the pointwise sum of two functions and the pointwise product of two functions respect the property. In this case, they either enjoy the property or preserve it as appropriate and the resulting theorems, *const\_eval\_thm*, *id\_eval\_thm*, *plus\_eval\_thm* and *times\_eval\_thm* record these facts. These together with a principle of induction over the set of polynomial functions show that any polynomial function may be represented by a sequence of coefficients.

We continue to record a few more technicalities and then to prove *comp\_poly\_thm*, which asserts that the functional composition of two polynomial functions is again a polynomial function.

The remaining theorems about polynomials build up to a proof that a polynomial function that is not identically zero has only finitely many roots (*poly\_roots\_finite\_thm*). This is proved using the special case of polynomial division in which the divisor is linear (which has the consequences familiarly known as the factor and remainder theorems).

HOL Constant

**Roots** :  $('a \rightarrow \mathbb{R}) \rightarrow 'a \text{ SET}$

---

$\forall f \bullet \text{Roots } f = \{x \mid f \ x = 0.\}$

*En route* we get useful facts about the interaction of polynomial evaluation with list operations and also a formula for the difference of two like powers which will turn out, fortuitously, to be just what is needed to prove the convergence of geometric series much later on.

Development of algebraic theory is not a main concern at this stage so we stop at this theorem. A later paper in this series of case studies will look at further results on roots, for example, Descartes' rule of signs. The next step might be to develop the theory of the degree of polynomials and of division, but that would probably be better done in general for polynomials over an arbitrary field.

Here is the complete list of the theorems discussed above. The first two theorems result from a manually conducted proof of the consistency of the definition of *PlusCoeffs*, which is not quite in the form that the infrastructure for automatic consistency proving can handle. The first theorem states the consistency of the definition in the format expected by the **ProofPower** infrastructure. We then adopt the convention of saving a theorem capturing the intended defining property, here *plus-coeffs-def*, which is derived automatically once the consistency proof has been done.

<i>PlusCoeffs_consistent</i>	<i>poly_func_eq_poly_eval_thm</i>	<i>poly_diff_powers_thm</i>
<i>plus-coeffs-def</i>	<i>const_poly_func_thm</i>	<i>length_plus-coeffs_thm</i>
<i>const_eval_thm</i>	<i>id_poly_func_thm</i>	<i>poly_linear_div_thm</i>
<i>id_eval_thm</i>	<i>plus_poly_func_thm</i>	<i>poly_remainder_thm</i>
<i>plus_eval_thm</i>	<i>times_poly_func_thm</i>	<i>poly_factor_thm</i>
<i>const_times_eval_thm</i>	<i>comp_poly_func_thm</i>	<i>poly_roots_finite_thm</i>
<i>times_eval_thm</i>	<i>poly_eval_append_thm</i>	
<i>poly_induction_thm</i>	<i>poly_eval_rev_thm</i>	

## 2.3 Algebraic Results

To deal with limits we will need a number of simple algebraic facts about the absolute value function, for example *ℝ\_abs\_plus\_thm*, which is the triangle inequality. Also, for dealing with power series we need some elementary facts about the operator  $x \hat{=} m$  that raises a real number to a natural number power. These facts and miscellaneous others are collected together here with a view to moving them into the theory of reals (and, perhaps, augmenting the linear arithmetic decision procedure to deal with absolute values). The names of the theorems are listed below:

<i>ℝ_0_≤_abs_thm</i>	<i>ℝ_¬_0_abs_thm</i>	<i>ℝ_ℕ_exp_recip_thm</i>
<i>ℝ_abs_plus_thm</i>	<i>ℝ_abs_≤_less_interval_thm</i>	<i>ℝ_ℕ_exp_recip_thm1</i>
<i>ℝ_abs_subtract_thm</i>	<i>ℝ_less_recip_less_thm</i>	<i>ℝ_ℕ_exp_1_≤_thm</i>
<i>ℝ_abs_plus_minus_thm</i>	<i>ℝ_plus_recip_thm</i>	<i>ℝ_ℕ_exp_less_1_mono_thm</i>
<i>ℝ_abs_diff_bounded_thm</i>	<i>ℕℝ_recip_thm</i>	<i>ℝ_ℕ_exp_less_mono_thm</i>
<i>ℝ_abs_plus_bc_thm</i>	<i>ℕℝ_0_less_recip_thm</i>	<i>ℝ_ℕ_exp_less_1_mono_thm1</i>
<i>ℝ_abs_abs_minus_thm</i>	<i>ℕℝ_recip_not_eq_0_thm</i>	<i>ℝ_ℕ_exp_linear_estimate_thm</i>
<i>ℝ_abs_abs_thm</i>	<i>ℝ_ℕ_exp_0_1_thm</i>	<i>ℝ_0_≤_square_thm</i>
<i>ℝ_abs_times_thm</i>	<i>ℝ_ℕ_exp_square_thm</i>	<i>ℝ_square_eq_0_thm</i>
<i>ℝ_abs_ℝ_ℕ_exp_thm</i>	<i>ℝ_ℕ_exp_0_less_thm</i>	<i>ℝ_bound_below_2_thm</i>
<i>ℝ_abs_eq_0_thm</i>	<i>ℝ_ℕ_exp_1_less_mono_thm</i>	<i>ℝ_bound_below_3_thm</i>
<i>ℝ_abs_≤_0_thm</i>	<i>ℝ_ℕ_exp_1_less_mono_thm1</i>	<i>ℝ_max_2_thm</i>
<i>ℝ_abs_0_thm</i>	<i>ℝ_≤_times_mono_thm</i>	<i>ℝ_max_3_thm</i>
<i>ℝ_abs_recip_thm</i>	<i>ℝ_ℕ_exp_¬_eq_0_thm</i>	<i>ℝ_min_2_thm</i>
<i>ℝ_abs_squared_thm</i>	<i>ℝ_ℕ_exp_plus_thm</i>	<i>ℝ_min_3_thm</i>
<i>ℝ_abs_less_times_thm</i>	<i>ℝ_ℕ_exp_times_thm</i>	

## 2.4 The Archimedean Property

It is a consequence of the completeness of the real numbers that they are an archimedean ordered ring: given any  $x, y > 0$  there is a natural number  $m$  such that  $y < mx$ . The following theorems

capture this property and some consequences. What we call the archimedean division theorem says that given  $d > 0$  and  $y \geq 0$ , there is a natural number  $q$  and a real number such that  $y = qd + r$  and  $0 \leq r < d$ .

*$\mathbb{R}$ \_archimedean\_thm*  
 *$\mathbb{R}$ \_archimedean\_recip\_thm*  
 *$\mathbb{R}$ \_archimedean\_times\_thm*

*$\mathbb{R}$ \_archimedean\_division\_thm*  
 *$\mathbb{R}$ \_N\_exp\_tends\_to\_infinity\_thm*  
 *$\mathbb{R}$ \_N\_exp\_tends\_to\_0\_thm*

## 2.5 Limits of Sequences of Numbers

We will write  $s \rightarrow x$  to indicate that the sequence  $s_n$ , indexed by natural numbers, tends to the limit  $x$  as  $n$  tends to infinity. We use real-valued functions on the natural numbers  $\mathbb{N}$  to represent such sequences. The definition is completely standard:

SML

```
| declare_infix(200, "->");
```

HOL Constant

```
| $-> : (N → R) → R → BOOL
```

---

```
|  $\forall s \ x \bullet s \rightarrow x \Leftrightarrow \forall e \bullet \mathbb{N} \ 0 < e \Rightarrow \exists n \bullet \forall m \bullet n \leq m \Rightarrow Abs(s \ m - x) < e$ 
```

The notions of closed and open intervals are used to make some of the results slightly more readable. The early part of the theory does not depend heavily on these notions. Later on we will develop some elementary topological results. The more interesting of these results, e.g., the Heine-Borel theorem depend on later definitions and results so they are collected together in section 2.10 below (with one or two honourable exceptions in section 2.8 which are needed earlier on).

HOL Constant

```
| ClosedInterval : R → R → R SET
```

---

```
|  $\forall x \ y \bullet \text{ClosedInterval } x \ y = \{t \mid x \leq t \wedge t \leq y\}$ 
```

HOL Constant

```
| OpenInterval : R → R → R SET
```

---

```
|  $\forall x \ y \bullet \text{OpenInterval } x \ y = \{t \mid x < t \wedge t < y\}$ 
```

New sequences may be formed from old using the field operations. If the original sequences converge then so do the new ones (providing we are careful to avoid dividing by zero). The first few theorems on limits capture this observation and some consequences, e.g., *poly\_lim\_seq\_thm*, which says that if  $s_m$  is a sequence converging to a limit,  $t$ , and  $f(x)$  is a polynomial, then the sequence  $f(s_m)$  converges to  $f(t)$ .

None of this theory would be very interesting if there were no convergent sequences, so we show that the sequence whose  $m$ -th term is  $x + \frac{1}{m+1}$  converges to  $x$  for any  $x$ . This is *lim\_seq\_recip\_N\_thm*. Here we are using a very common and useful device to avoid talking about division by zero: if  $m$  is

a natural number, then  $m + 1$  is a non-zero natural number and it is often easier just to state results about  $m + 1$  rather than have hypotheses of the form  $0 < m$ .

As it will turn out, sequences are very important in our approach to limiting processes generally, so we develop the necessary theory a little further here. We show, for example, in *lim\_seq\_¬\_eq\_thm* that any real  $x$  can be given as the limit of a sequence  $s_m$  such that no  $s_m = x$ . We also show, very importantly, in *lim\_seq\_unique\_thm*, that if a sequence converges to a limit then that limit is unique. A similar uniqueness property will be derived for all of the limiting processes we investigate.

<i>const_lim_seq_thm</i>	<i>recip_lim_seq_thm</i>	<i>lim_seq_¬_eq_thm1</i>
<i>plus_lim_seq_thm</i>	<i>lim_seq_choice_thm</i>	<i>lim_seq_¬_0_thm</i>
<i>lim_seq_bounded_thm</i>	<i>abs_lim_seq_thm</i>	<i>lim_seq_upper_bound_thm</i>
<i>times_lim_seq_thm</i>	<i>lim_seq_recip_ℕ_thm</i>	<i>lim_seq_unique_thm</i>
<i>minus_lim_seq_thm</i>	<i>lim_seq_¬_eq_thm</i>	<i>lim_seq_diffs_tend_to_0_thm</i>
<i>poly_lim_seq_thm</i>	<i>lim_seq_shift_thm</i>	

## 2.6 Continuity

Now we give the usual  $\epsilon$ - $\delta$  definition for continuity of a function,  $f$  at a point  $x$ . Both kinds of  $\epsilon$  are tied up in the ProofPower-HOL library for something else (viz. Hilbert's choice operator and set-membership). We therefore just use  $e$  and  $d$ . Continuity is formulated as an infix relation between a function and the point at which its continuity is asserted.

SML

```
| declare_infix(200, "Cts");
```

HOL Constant

```
| $Cts : (ℝ → ℝ) → ℝ → BOOL
```

---

```
| ∀f x• f Cts x
```

```
| ⇔   ∀e•ℕR 0 < e ⇒ ∃d•ℕR 0 < d ∧ ∀y• Abs(y - x) < d ⇒ Abs(f y - f x) < e
```

As observed in [2], several notions of limit arise and it is desirable to have common ways of dealing with them. Harrison's approach is via the general notion of convergence nets. We choose the more homely notion of sequential convergence as our way around this problem.

As we shall state and prove later, to say that  $f$  is continuous at  $x$  is to say that  $f(y)$  tends to  $f(x)$  as  $y$  tends to  $x$ . As the first of our theorems, *cts\_lim\_seq\_thm*, shows, this holds iff. for any sequence  $s_m$  whose limit is  $x$ , we have that the sequence  $f(s_m)$  tends to  $f(x)$ .

This approach makes the proofs of the continuity of functions formed from continuous functions by functional composition and by applying field-operations trivial applications of analogous results on sequences. I have not seen this useful observation made in any elementary textbook on the subject, the reasons, I suspect, being two-fold: (i) proving these results about continuity from the definitions is considered to be a good exercise for the student, and, (ii), the analogue of this approach does not work in an arbitrary topological space (although spaces in which it fails are fairly exotic). Nonetheless, it works very well for us here, so well, in fact, that we spend the time to prove a couple of variants of the result on continuity and limits of sequences which let us restrict attention to sequences which are confined to a given neighbourhood of the point at which continuity is asserted and which

avoid that point. This immediately shows that continuity is a local property: if two functions,  $f$  and  $g$ , agree in some open interval containing  $x$  and  $g$  is continuous at  $x$ , then so is  $f$ .

At this point in the proof script, we introduce a few simple proof procedures to help with applying the results on the continuity of algebraic combinations of continuous functions. In practice, we will want to prove that specific combinations are continuous, e.g.,  $1 + 2f(x)$ , given the continuity of  $f$ . However, it requires (a very special case of) higher-order matching to apply the theorems to the function  $\lambda x \bullet 1 + 2f(x)$ . The proof procedures comprise: (a) a conversion to help put in the necessary  $\beta$ -redexes, e.g., to rewrite  $\lambda x \bullet 1 + 2f(x)$  as  $\lambda x \bullet (\lambda x \bullet 1)(x) + (\lambda x \bullet 2f(x))(x)$  so that it matches the statement of *plus\_cts\_thm*; and, (b), a tactic, *simple\_cts\_tac*, which selects the right theorem to apply and uses the conversion to apply it.

This heuristic for automatically proving continuity is very useful, but the implementation is very simple-minded (for example, it does not allow the user to extend the range of functions supported). Later on, we will give a more sophisticated analogue for calculating derivatives. We choose to defer a more sophisticated approach to automatic continuity proving to separate work on topology, since in any case, the particular functions we are interested in in this document turn out to be continuous by dint of being differentiable.

Once the basic facts about continuity of algebraic combinations of continuous functions are in place, we prove a few theorems for later use, for example, in reasoning about compact sets. Compact sets are defined in terms of open coverings and so it is useful to have the characterisation of continuity in terms of open sets. Following our approach of taking sequential convergence as basic, we first characterise sequential convergence in terms of open sets and then use that characterisation to derive one for continuity.

Next we prove a uniqueness property for continuity: namely, that if  $f$  is continuous at  $x$ , then  $f(x)$  is uniquely determined by the values  $f(y)$  where  $y \neq x$  ranges over any given neighbourhood of  $x$ . To do this, we first show that continuity of  $f$  at  $x$  implies a principle for estimating  $f(x)$  (*cts\_estimate\_thm*) and as a consequence a principle for showing that  $f(x) = 0$  (*cts\_estimate\_0\_thm*). This latter principle applied to the difference of two functions  $f$  and  $g$  which are continuous and agree in some neighbourhood of  $x$ , except perhaps at  $x$ , shows that two such functions must agree at  $x$ , which is the uniqueness principle we want.

The final result in this section on continuity states that a function which is both continuous in the sense of Darboux (i.e., it satisfies the conclusion of the intermediate value theorem) and monotonic increasing is continuous in the usual sense. This is preceded by a result which reduces the number of cases that need to be considered in proving the result on Darboux continuity: it says that if  $f(x)$  lies in the open interval  $(a, b)$ , then when you test  $f$  for continuity at  $x$  using open intervals, you can assume the end-points of the “challenge” interval,  $(c, d)$  are both contained in  $(a, b)$ . The result on Darboux continuity will be used to show that the natural logarithm function is continuous.

*cts\_lim\_seq\_thm*  
*cts\_lim\_seq\_thm1*  
*cts\_lim\_seq\_thm2*  
*cts\_local\_thm*  
*const\_cts\_thm*  
*id\_cts\_thm*  
*plus\_cts\_thm*  
*times\_cts\_thm*  
*poly\_cts\_thm*  
*ℝ\_ℕ\_exp\_cts\_thm*

*comp\_cts\_thm*  
*minus\_cts\_thm*  
*minus\_comp\_cts\_thm*  
*recip\_cts\_thm*  
*recip\_comp\_cts\_thm*  
*ℝ\_ℕ\_exp\_comp\_cts\_thm*  
*abs\_cts\_thm*  
*abs\_comp\_cts\_thm*  
*cts\_extension\_thm1*  
*cts\_extension\_thm*

*open\_ℝ\_delta\_thm*  
*lim\_seq\_open\_ℝ\_thm*  
*cts\_open\_ℝ\_thm*  
*closed\_interval\_closed\_thm*  
*cts\_estimate\_thm*  
*cts\_estimate\_0\_thm*  
*cts\_limit\_unique\_thm*  
*cts\_open\_interval\_thm*  
*darboux\_cts\_mono\_thm*  
*darboux\_cts\_mono\_thm1*

## 2.7 Derivatives

We now turn to derivatives and give the standard  $\epsilon$ - $\delta$  definition for the derivative of a function,  $f$  at a point  $x$ . The notion of a derivative as formulated here is really a ternary relation: “ $f$  has derivative  $c$  at  $x$ ”. We represent this as the infix operation between  $f$  and  $c$  whose value is the propositional function that characterises the values  $x$  for which the derivative  $f$  is  $c$ .

SML

```
| declare_infix(200, "Deriv");
```

HOL Constant

```
| $Deriv : (ℝ → ℝ) → ℝ → ℝ → BOOL
```

---

```
|
|  $\forall f\ c\ x \bullet$ 
|    $(f\ Deriv\ c)\ x \Leftrightarrow$ 
|    $\forall e \bullet \text{NIR } 0 < e \Rightarrow \exists d \bullet$ 
|      $\text{NIR } 0 < d$ 
|    $\wedge \forall y \bullet Abs(y - x) < d \wedge \neg y = x \Rightarrow Abs((f\ y - f\ x)/(y - x) - c) < e$ 
```

As usual, we will investigate the interaction of the field operations and functional composition with derivatives and use the results to say something about derivatives of polynomials. In order to state the result about polynomials, we need the following definition, which gives the operation on lists of coefficients that corresponds to taking the derivative of a polynomial function.

HOL Constant

```
| DerivCoeffs : ℝ LIST → ℝ LIST
```

---

```
|
|    $DerivCoeffs\ [] = []$ 
|  $\wedge (\forall c\ l \bullet DerivCoeffs\ (Cons\ c\ l) = PlusCoeffs\ l\ (Cons\ (\text{NIR } 0)\ (DerivCoeffs\ l)))$ 
```

Following Harrison, we make much use of Carathéodory’s characterisation of the derivative in terms of the existence of a continuous function satisfying certain conditions. This turns all the statements about derivatives into statements about continuity which are easy to prove once the necessary witnesses are supplied.

After the results about derivatives of algebraic combinations of functions and polynomials, we show that the derivative is a local property and that the derivative is unique. For later use, we also provide the characterisation of derivatives in terms of limits of function values.

These results are followed in the proof script by the coding of an inference rule, and a corresponding tactic that automatically calculate derivatives of algebraic combinations of functions whose derivatives are known. Unlike the tactic for continuity, these automatic proof procedures are fairly general — they are parameterised by a list of theorems giving facts about the derivatives. The algorithm used is intended to be a little more general than the one described by Harrison; in particular, the algorithm is formulated so that the chain rule can be presented as one of the theorems rather than being wired in.

<i>caratheodory_deriv_thm</i>	<i>times_deriv_thm</i>	<i>recip_deriv_thm</i>
<i>deriv_cts_thm</i>	<i>times_const_deriv_thm</i>	<i>recip_comp_deriv_thm</i>
<i>const_deriv_thm</i>	<i>comp_deriv_thm</i>	<i>poly_deriv_thm</i>
<i>id_deriv_thm</i>	<i>minus_deriv_thm</i>	<i>deriv_local_thm</i>
<i>plus_deriv_thm</i>	<i>minus_comp_deriv_thm</i>	<i>deriv_lim_fun_thm</i>
<i>plus_const_deriv_thm</i>	<i>ℝ_N_exp_deriv_thm</i>	<i>deriv_unique_thm</i>

Note: the statement of *deriv\_lim\_fun\_thm* uses the notion of the limit of a function value introduced in section 2.12 below.

## 2.8 Some Classical Theorems

We proceed to prove our selection of the classical miscellany of extremely fruitful facts about continuous and differentiable functions. The selection was motivated both bottom-up, by picking results for their intrinsic interest, and top-down, by picking results that will be needed later. One might expect to find l’Hôpital’s rule in this section, but to state it properly requires some additional notions that we prefer to define later, see section 2.16.

The notions of open, closed and compact sets are useful in stating some of these results.

HOL Constant

***Open<sub>R</sub>*** : ℝ SET SET

---

*Open<sub>R</sub>* = {A | ∀t•t ∈ A ⇒ ∃x y•t ∈ *OpenInterval* x y ∧ *OpenInterval* x y ⊆ A}

[Note that  $\sim$  below is not arithmetic negation, but the operation of complementing a set with respect to the universe of its type.]

HOL Constant

***Closed<sub>R</sub>*** : ℝ SET SET

---

*Closed<sub>R</sub>* = {A |  $\sim$  A ∈ *Open<sub>R</sub>*}

HOL Constant

***Compact<sub>R</sub>*** : ℝ SET SET

---

*Compact<sub>R</sub>* =  
 {A | ∀ V• V ⊆ *Open<sub>R</sub>* ∧ A ⊆ ∪ V ⇒ ∃W• W ⊆ V ∧ W ∈ *Finite* ∧ A ⊆ ∪ W}

The first of our collection of “classical” theorems, which we call the “curtain induction” principle, is actually one that I have never seen stated before! However, it is very useful in proving Bolzano’s principle of bisection and proving that (bounded) closed intervals are compact. It says that if a property  $P$  is such that  $P(x)$  entails  $P(y)$  for every  $y < x$  and if for any  $x$ , there is a  $y < x$  such that  $P(y)$  entails  $P(z)$  for some  $z > x$ , then if  $P$  holds anywhere it holds everywhere. One thinks of sliding a curtain from left to right across the real line starting at the given point where  $P$  is known to hold. This captures the essence of one kind of supremum argument in a fairly neat way.

In general, when dealt with fully formally, reasoning about suprema is more slippery than one might think — a problem that is slightly exacerbated by some small infelicities in the quantifier structure

of some of the theorems about suprema in the theory  $\mathbb{R}$ . We therefore do very little direct reasoning about suprema in the proofs. The problems with the statements about suprema in the theory  $\mathbb{R}$  are purely technical and could be remedied (they just sometimes fail to work nicely with some of the standard proof procedures). However, the alternative approaches adopted here work so much better that I have felt little pressure to fix them.

With the curtain induction principle and one or two other lemmas, e.g., that the union of a finite chain of sets is actually a member of the chain, we can then reel off the fairly standard proofs of Bolzano's bisection principle, the fact that continuous functions are bounded and attain their maximum values in closed intervals (more generally in compact sets), the intermediate value theorem, the mean value theorem etc.

The mean value theorem appears here in three guises. *cauchy\_mean\_value\_thm* is the general statement about two functions  $f$  and  $g$  continuous in a closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$  that Cauchy proved as his mean value theorem. This is the result we need later to prove l'Hôpital's rule. *mean\_value\_thm* is the usual statement about a single function  $f$  satisfying the same conditions as in *cauchy\_mean\_value\_thm*. Finally *mean\_value\_thm1* is the weaker variant where  $f$  is required to be differentiable on  $[a, b]$  which is often useful in practice (because that is the hypothesis one has to hand). In each of these mean value theorems, our statement of the theorem is what is obtained from the traditional statement by cross-multiplying to eliminate division in favour of multiplication. This is convenient in practice and brings out a symmetry in *cauchy\_mean\_value\_thm* (which is actually slightly more general as a result).

By contrast with Harrison's approach, later on, when we need explicit numerical estimates, for example, in demonstrating the existence of  $\pi$ , we tend to use theorems like the mean value theorem, rather than obtaining numerical estimates from power series. *deriv\_linear\_estimate\_thm* is an example of this kind of estimation.

<i>curtain_induction_thm</i>	<i>intermediate_value_thm</i>	<i>mean_value_thm1</i>
<i>bisection_thm</i>	<i>local_min_thm</i>	<i>deriv_0_thm1</i>
<i>closed_interval_compact_thm</i>	<i>local_max_thm</i>	<i>deriv_0_thm</i>
<i>finite_chain_thm</i>	<i>rolle_thm</i>	<i>deriv_0_thm2</i>
<i>cts_compact_maximum_thm</i>	<i>cauchy_mean_value_thm</i>	<i>deriv_0_less_thm</i>
<i>cts_maximum_thm</i>	<i>mean_value_thm</i>	<i>deriv_linear_estimate_thm</i>
<i>cts_abs_bounded_thm</i>		

## 2.9 Excursus: Cantor's Theorem

The next block of theorems is an excursion. We prove Cantor's theorem that the reals are uncountable. The proof is via nested intervals (and generalises to show that any perfect set in a metric space is uncountable; see Rudin [5]). This material does feed in, in spirit, if not in actual use of the theorems, to the theory of Cauchy sequences.

$\mathbb{R}$ _mono_inc_seq_thm	<i>nested_interval_diag_thm</i>
$\mathbb{R}$ _mono_dec_seq_thm	<i>nested_interval_intersection_thm</i>
<i>nested_interval_bounds_thm</i>	$\mathbb{R}$ _uncountable_thm



## 2.10 Some Topology

In this section we show that the set of open and closed sets as we have defined them do indeed satisfy the rules for the open and closed sets of a topological space. We then provide a selection of additional examples of open and closed sets. More interestingly, we prove that compact sets are closed and contain their maximum and minimum elements (if they are non-empty). The empty set is compact, but the whole real line is not. Closed subsets of compact sets are compact. A set is compact iff. it is closed and bounded (the Heine-Borel theorem). Finally,  $\mathbb{R}$  is topologically connected, i.e., it cannot be represented as the union of two disjoint open sets.

<i><math>\cap</math>_open_interval_thm</i>	<i>complement_closed_interval_thm</i>
<i><math>\cap</math>_closed_interval_thm</i>	<i>half_infinite_intervals_open_thm</i>
<i><math>\cup</math>_open_ℝ_thm</i>	<i>half_infinite_intervals_closed_thm</i>
<i><math>\cup</math>_open_ℝ_thm</i>	<i>empty_universe_open_closed_thm</i>
<i><math>\cap</math>_open_ℝ_thm</i>	<i>compact_closed_thm</i>
<i><math>\cap</math>_closed_ℝ_thm</i>	<i>compact_min_max_thm</i>
<i><math>\cap</math>_closed_ℝ_thm</i>	<i>closed_⊆_compact_thm</i>
<i><math>\cup</math>_closed_ℝ_thm</i>	<i>empty_universe_compact_thm</i>
<i>open_interval_open_thm</i>	<i>heine_borel_thm</i>
<i>complement_open_interval_thm</i>	<i>ℝ_connected_thm</i>

## 2.11 Cauchy Sequences

Up to this point, we have generally only been able to show that particular sequences converge when a formula for the limit is known. In this section we prove that a sequence is convergent iff. it is a Cauchy sequence. The argument is the standard one: that a convergent sequence is necessarily a Cauchy sequence is a straightforward consequence of the definition of the limit of a sequence; conversely, a Cauchy sequence is bounded and hence has a  $\limsup$  and a  $\liminf$ , and by the Cauchy property, these two must be equal, and their common value is the limit of the sequence. Rather than define the notions of  $\limsup$  and  $\liminf$  as constants in the theory, we simply give theorems that assert the existence of these values.

*En passant*, we prove one or two useful results about monotonic increasing and decreasing sequences.

<i>lim_seq_cauchy_seq_thm</i>	<i>lim_seq_mono_inc_sup_thm</i>	<i>lim_inf_thm</i>
<i>fin_seq_bounded_thm</i>	<i>lim_seq_mono_inc_thm</i>	<i>cauchy_seq_lim_seq_thm</i>
<i>cauchy_seq_bounded_above_thm</i>	<i>lim_seq_mono_dec_thm</i>	<i>lim_seq_↔_cauchy_seq_thm</i>
<i>cauchy_seq_bounded_below_thm</i>	<i>lim_sup_thm</i>	

## 2.12 Limits of Function Values

We will write  $(f \dashrightarrow c)x$  to indicate that  $f(t)$  tends to the limit  $c$  as  $t$  tends to  $x$ :

SML

```
|declare_infix(205, "dashrightarrow");
```

HOL Constant

$\$--> : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \text{BOOL}$

$\forall f \ c \ x \bullet$

$(f \ --> \ c) \ x$

$\Leftrightarrow \forall e \bullet \text{NIR } 0 < e$

$\Rightarrow \exists d \bullet \text{NIR } 0 < d \wedge \forall y \bullet \text{Abs}(y - x) < d \wedge \neg y = x \Rightarrow \text{Abs}(f \ y - c) < e$

As usual, we prove the standard facts about the interaction between limits of function values and the field operations. These are all proved using a characterisation of the function value notion of limit in terms of limits of sequences. We then show that limits of function values are unique if they exist and give two results about limits of function values and functional composition: the first of these is the obvious one: if  $f$  tends to  $t$  at  $x$  and  $g$  is continuous at  $t$ , then  $\lambda x \bullet g(fx)$  tends to  $g(t)$  at  $x$ ; the second is less obvious: if  $f$  is continuous at  $x$  and one-to-one on some open interval containing  $x$ , and if  $g$  tends to  $t$  at  $f(x)$ , then  $\lambda x \bullet g(fx)$  tends to  $t$  at  $x$ . I have not seen the latter result in the textbooks, but it seems to be needed to show that if  $g$  is right-inverse to  $f$  in some open interval  $(a, b)$  and if  $f$  has derivative  $c \neq 0$  at  $g(x)$  for  $x$  in  $(a, b)$ , then  $g$  has derivative  $\frac{1}{c}$  at  $x$ . This is needed later to calculate the derivative of the natural logarithm function.

*lim\_fun\_lim\_seq\_thm*

*const\_lim\_fun\_thm*

*id\_lim\_fun\_thm*

*plus\_lim\_fun\_thm*

*times\_lim\_fun\_thm*

*recip\_lim\_fun\_thm*

*comp\_lim\_fun\_thm*

*comp\_lim\_fun\_thm1*

*poly\_lim\_fun\_thm*

*lim\_fun\_upper\_bound\_thm*

*lim\_fun\_unique\_thm*

*lim\_fun\_local\_thm*

*deriv\_lim\_fun\_thm1*

*comp\_lim\_fun\_thm2*

*cts\_lim\_fun\_thm*

*inverse\_deriv\_thm*

## 2.13 Limits of Sequences of Functions

We write  $(u \text{--->} h)X$  to indicate that the sequence of functions  $u_n$ , indexed by natural numbers, tends uniformly to the limit function  $h$  on the set  $X$  as  $n$  tends to infinity.

SML

$\text{declare\_infix}(205, \text{--->});$

HOL Constant

$\$ \text{--->} : (\mathbb{N} \rightarrow \mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \text{ SET} \rightarrow \text{BOOL}$

$\forall u \ h \ X \bullet$

$(u \ \text{--->} \ h) \ X$

$\Leftrightarrow \forall e \bullet \text{NIR } 0 < e \Rightarrow \exists n \bullet \forall m \ y \bullet n \leq m \wedge y \in X \Rightarrow \text{Abs}(u \ m \ y - h \ y) < e$

We prove a small selection of results about algebraic combinations of uniform limits of sequences of functions, just covering constants, addition and multiplication. The proofs are done, essentially, by copy-and-paste reuse of the analogous results for sequences of values. Note that we could have defined the notion of the limit of a sequence of values as a special case of a limit of a sequence of functions (namely as a limit of a sequence of constant functions). However, it seems clearer not to do

this (and some facts about sequences of values do not hold in general, e.g., if  $f_n$  converges uniformly in  $X$  as  $n$  tends to  $\infty$ ,  $\frac{1}{f_n}$  is not uniformly convergent in general).

We then show that (uniformly) convergent sequences of functions are Cauchy sequences and, more interestingly, that sequences of functions that are uniformly Cauchy are uniformly convergent. This is followed by our two big theorems on uniform convergence based on Rudin [5]: the first of these is a version of Rudin's Theorem 7.11, which is an interchange theorem relating our three notions of limit. More precisely, our formulation says that if  $u_m$  converges uniformly to  $f$  on some set of the form  $(a, b) \setminus \{x\}$  for  $x$  in the open interval  $(a, b)$  and if for each  $m$   $u_m(y)$  converges to  $s_m$  as  $y$  tends to  $x$ , then  $f(y)$  converges to some limit  $c$  as  $y$  tends to  $x$  and  $s_m$  converges to that same value  $c$  as  $m$  tends to  $\infty$ . Rudin states and proves this in greater generality, but we just do the case that is needed for the the second big theorem, namely Rudin's Theorem 7.17.

The second big theorem says, in essence, that under suitable conditions, the derivative of a limit of a sequence of functions exists and may be calculated as the limit of the derivatives of the functions in the sequence. In our formulation, the theorem says that if, a sequence of functions  $du_m$  converges uniformly to some function  $df$  on an open interval  $(a, b)$  and each  $du_m$  is the derivative of some function  $u_m$  throughout  $(a, b)$ , and if the sequence of values  $u_m(x_0)$  is convergent for some  $x_0$  in  $(a, b)$ , then the sequence of functions  $u_m$  is uniformly convergent to a function  $f$  in  $(a, b)$  whose derivative is given by  $df$  throughout  $(a, b)$ . This is the key result that justifies "differentiation under the summation sign" for convergent power series.

Our formulation of the result on uniform convergence and derivatives differs from the standard one (as given in Rudin) in using open intervals rather than closed ones. It is traditional to state results about uniform convergence for closed intervals, or more generally for compact sets. Unfortunately, to get this to work requires some special treatment of the end-points of the interval (see Rudin's Definition 5.1 for the sort of things that go on). We prefer to stick with the single notion of a two-sided derivative. This does mean that some results might have to be stated as holding for all open intervals (or sets) contained in some closed interval (or compact set) rather than holding on the closed interval or compact set. However, this has not arisen in practice, and there is no real loss of generality: any function that is, say, differentiable, on a closed interval  $[a, b]$  can be extended to a differentiable function on an open interval  $(A, B)$  containing  $[a, b]$  and our way of formulating the kind of results in question will then apply to  $(A, B)$ .

It is instructive to compare the formal treatment of these two big theorems with their treatment as in Rudin: the formal versions make it much more evident that these results are pure existence theorems, with proofs that conjure up witnesses out of the air using the Cauchy condition and use uniqueness properties of limits to show that these witnesses satisfy the requirements. The traditional mathematical notation, in contrast, gives an illusion of "constructiveness", as if notations like  $\lim_{n \rightarrow \infty} f'_n(x)$  denoted things that exist *a priori*.

It is also noteworthy that these two theorems and the root test for convergence of power series allow one to develop the theory of the transcendental functions via power series without using integration. As noted by Hanne Gottlieb [1], many text books bypass these theorems using results about integration to prove that power series can be differentiated term by term. As shown here, this is not necessary if you prove the more general results.

*const\_unif\_lim\_seq\_thm*  
*plus\_unif\_lim\_seq\_thm*  
*bounded\_unif\_limit\_thm*  
*times\_unif\_lim\_seq\_thm*  
*unif\_lim\_seq\_bounded\_thm*  
*unif\_lim\_seq\_⊆\_thm*  
*unif\_lim\_seq\_shift\_thm*  
*unif\_lim\_seq\_cts\_thm*

*unif\_lim\_seq\_cauchy\_seq\_thm*  
*cauchy\_seq\_unif\_lim\_seq\_thm*  
*unif\_lim\_seq\_pointwise\_lim\_seq\_thm*  
*unif\_lim\_seq\_pointwise\_unique\_thm*  
*unif\_lim\_seq\_unique\_thm*  
*lim\_fun\_lim\_seq\_interchange\_thm*  
*unif\_lim\_seq\_deriv\_estimate\_thm*  
*unif\_lim\_seq\_deriv\_thm*

## 2.14 Series and Power Series

The *series* defined by a sequence of real terms is, to all intents and purposes, the sequence of partial sums of the sequence.

HOL Constant

$\mathbf{Sigma} : (\mathbb{N} \rightarrow \mathbb{R}) \rightarrow (\mathbb{N} \rightarrow \mathbb{R})$	
$(\forall s \bullet \mathbf{Sigma} \ s \ 0 = \mathbb{NR} \ 0)$	
$\wedge (\forall s \ n \bullet \mathbf{Sigma} \ s \ (n+1) = \mathbf{Sigma} \ s \ n + s \ n)$	

The *power series* defined by a sequence of real coefficients is, in a similar vein, the sequence of polynomial functions whose coefficients are given by leading subsequences of the sequence:

HOL Constant

$\mathbf{PowerSeries} : (\mathbb{N} \rightarrow \mathbb{R}) \rightarrow (\mathbb{N} \rightarrow \mathbb{R} \rightarrow \mathbb{R})$	
$\forall s \ n \bullet \mathbf{PowerSeries} \ s \ n = \mathbf{PolyEval} \ (s \ To \ n)$	

### 2.14.1 Elementary Properties

We first prove a few elementary properties of series and power series.

<i>power_series_series_thm</i>	<i>series_0_thm</i>	<i>lim_series_shift_thm</i>
<i>series_power_series_thm</i>	<i>const_times_series_thm</i>	<i>lim_series_shift_∃_thm</i>
<i>power_series_0_arg_thm</i>	<i>lim_seq_ℕℝ_recip_eq_0_thm</i>	<i>lim_series_bounded_thm</i>
<i>series_power_limit_0_thm</i>	<i>lim_seq_ℝℕ_exp_eq_0_thm</i>	
<i>plus_series_thm</i>	<i>series_shift_thm</i>	

### 2.14.2 Geometric Series

We next show that the geometric series  $\sum x^n$  has radius of convergence 1, using the result on the polynomial  $x^n - y^n$  previously proved (as *poly\_diff\_powers\_thm*) to give the explicit formula for the limit.

*geometric\_sum\_thm*  
*geometric\_series\_thm*

*geometric\_series\_series\_thm*  
*geometric\_series\_thm1*

### 2.14.3 Convergence Tests

We now give the standard convergence tests for series and power series. The first two results are a good example of a case where our *idiot savant* has no need of spoon feeding: the Weierstrass test for uniform convergence of the sum of a series of functions has the comparison test for convergence of series of values as an immediate corollary of the special case when the functions are constant. There is no point in presenting the comparison test first (as is invariably done in the text books).

Our formulation of the root test for the convergence of a series is somewhat non-standard. This test is usually couched in terms of  $\limsup \sqrt[n]{|s_n|}$ , but we have chosen not to introduce the theory of  $n$ -th roots at this point in the development, nor to define a constant for the notion of *limes superior*. However, the usual proof of the root test immediately derives from the usual statement the fact that a series for which  $\limsup \sqrt[n]{|s_n|}$  exists and is less than 1 is majorised in absolute value by a convergent geometric series. We just make that our condition: this appears to be entirely equivalent to the standard definition (and states the result in terms of what you generally have to prove to show the  $\limsup$  in question exists as required). With this formulation, the test will actually work up to a multiplicative constant, which can sometimes ease the burden of proof (see the statement of *root\_test\_thm*).

We then give the ratio test, firstly in its most general form which says that a series  $s_n$  of non-zero terms will converge if the ratios  $|\frac{s_{n+1}}{s_n}|$  are bounded above by some  $b < 1$  for all large enough  $n$ . We then give the more commonly stated form which requires these ratios to converge to a limit less than 1, which is strictly less general, but useful if you have to hand facts about the limits in question.

*weierstrass\_test\_thm*  
*comparison\_test\_thm*  
*simple\_root\_test\_thm*

*root\_test\_thm*  
*ratio\_test\_thm1*  
*ratio\_test\_thm*

### 2.14.4 Convergence of Power Series

We now give the main result on convergent power series which says that if a power series  $\sum c_n x^n$  converges pointwise in absolute value for  $x = B$  for some positive  $B$ , then for any positive  $b < B$ , the power series converges (viewed as a sequence of functions) uniformly for  $x$  in the open interval  $(-b, b)$  and is differentiable on that interval with a derivative given as the limit of the result of differentiating the power series term by term (the existence of this limit and the uniform convergence to it being part of the conclusion of the theorem).

After a few lemmas which break out some important steps on the main result, we present the main result in several guises, including a corollary about second derivatives that will be useful for the sine and cosine functions.

*power\_series\_convergence\_thm*  
*power\_series\_deriv\_coeffs\_thm*  
*power\_series\_deriv\_lemma1*  
*power\_series\_deriv\_lemma2*  
*power\_series\_scale\_arg\_thm*  
*power\_series\_scale\_coeffs\_thm*

*power\_series\_deriv\_lim\_seq\_convergence\_thm*  
*power\_series\_deriv\_convergence\_thm*  
*power\_series\_main\_thm*  
*power\_series\_main\_thm1*  
*power\_series\_main\_thm2*  
*power\_series\_main\_thm3*

## 2.15 Some Special Functions

In this section, we define the transcendental functions,  $\exp$ ,  $\log$ ,  $\sin$  and  $\cos$ . For geometric applications, it seems most natural to specify  $\exp$ ,  $\sin$  and  $\cos$  by differential equations. We will use power series to provide witnesses to the consistency of these specifications (i.e., to provide solutions to the differential equations). We will then rely on the differential equations to derive the main properties of the functions (including the fact that  $\exp$  is one-to-one, which justifies a specification of  $\log$  as its left inverse).

Before defining the transcendental functions, we introduce the factorial function, which we will need to give the terms of the power series that provide the witnesses. We make this a postfix operator with precedence higher than all the binary arithmetic operators.

SML

```
| declare_postfix(330, "!");
```

HOL Constant

$\mathbf{\$!} : \mathbb{N} \rightarrow \mathbb{N}$	
$0 ! = 1$ $\wedge (\forall m \bullet (m+1)! = (m+1) * m!)$	

Using the factorial function, we can state Taylor's theorem. While this theorem is not used in our approach to the transcendental functions, it is proved here for completeness, using one or two lemmas about the factorial function and a generalisation to  $n$ -th derivatives of Rolle's theorem.

*factorial\_linear\_estimate\_thm*    *factorial\_times\_recip\_thm*    *taylor\_thm*  
*factorial\_0\_less\_thm*            *gen\_rolle\_thm*

### 2.15.1 The Exponential Function

As promised, we define the exponential function by a differential equation with an initial condition.

HOL Constant

$\mathbf{Exp} : \mathbb{R} \rightarrow \mathbb{R}$	
$Exp \text{ (NR } 0) = \text{NR } 1$ $\wedge (\forall x \bullet (Exp \text{ Deriv } Exp \ x) \ x)$	

The first block of theorems lead up to the consistency of the above definition using the usual power series to provide the witness. Then we give a uniqueness property for the exponential function and develop its usual algebraic properties (viz., that `exp` is a strictly order-preserving homomorphism from the group of all real numbers under addition to the group of positive real numbers under multiplication).

We develop the properties of `exp` using an “axiomatic” approach along the lines of that given by Mitchell [4] for the sine and cosine functions. I am grateful to John Harrison for pointing out a considerable simplification to an earlier version of this argument.

Assume we have a function  $e$  satisfying the differential equations and initial conditions specified for `exp`. I.e.,  $e(0) = 1$ ,  $e$  is everywhere differentiable, and the derivative  $e'(x)$  is equal to  $e(x)$  for all  $x$ . For arbitrary  $y$ , put  $q_y(x) = e(x+y) \times e(-x)$ , then  $q_y$  is everywhere differentiable with  $q'_y(x) = 0$  and so  $q_y$  is constant. Since  $q_y(0) = e(y)$ , we must have  $e(x+y) \times e(-x) = e(y)$  for all  $x$  and  $y$ . Putting  $y = 0$ , this gives that  $e(x) \neq 0$  with  $e(x)^{-1} = e(-x)$  for all  $x$ . So  $e(x+y) = e(x) \times q_y(x) = e(x) \times e(y)$  for all  $x$  and  $y$ . Since  $e(0)$  is positive and  $e(x) \neq 0$  for any  $x$ , by the intermediate value theorem,  $e(x)$  is always positive.

The above shows that `exp` is indeed a homomorphism from the group of all real numbers under addition to the group of real numbers under multiplication. That this homomorphism is order-preserving follows from `deriv_0_less_thm` and the differential equation for `exp`, which we now know implies that the derivative of `exp` is everywhere positive.

<i>factorial_linear_estimate_thm</i>	<i>exp_def</i>	<i>exp_plus_thm</i>
<i>factorial_0_less_thm</i>	<i>exp_unique_thm</i>	<i>exp_clauses</i>
<i>factorial_times_recip_thm</i>	<i>exp_cts_thm</i>	<i>exp_less_mono_thm</i>
<i>exp_series_convergence_thm</i>	<i>exp_not_eq_0_thm</i>	<i>exp_power_series_thm</i>
<i>exp_consistency_thm</i>	<i>exp_minus_thm</i>	
<i>Exp_consistent</i>	<i>exp_0_less_thm</i>	

### 2.15.2 The Natural Logarithm Function

The natural logarithm function is defined as the left inverse of the exponential function.

HOL Constant

<b>Log</b> : $\mathbb{R} \rightarrow \mathbb{R}$
$\forall x \bullet \text{Log} (\text{Exp } x) = x$

We know enough about `exp` to show that it does indeed have a left inverse (which amounts to showing that it is one-to-one, which follows since it is strictly order-preserving). From this follow standard algebraic facts about `log` as well as the fact that it is continuous and a calculation of its derivative.

<i>Log_consistent</i>	<i>exp_0_less_onto_thm</i>	<i>log_deriv_thm</i>
<i>log_def</i>	<i>exp_log_thm</i>	<i>log_clauses</i>
<i>exp_one_one_thm</i>	<i>log_less_mono_thm</i>	<i>ℝ_ℕ_exp_exp_log_thm</i>
<i>exp_ℝ_ℕ_exp_thm</i>	<i>log_cts_thm</i>	

### 2.15.3 The Sine and Cosine Functions

The sine and cosine functions are defined by differential equations and initial conditions.

HOL Constant

	<b><i>Sin Cos</i></b> : $\mathbb{R} \rightarrow \mathbb{R}$
<hr/>	
	<i>Sin</i> (NR 0) = NR 0
$\wedge$	<i>Cos</i> (NR 0) = NR 1
$\wedge$	$(\forall x \bullet (\textit{Sin Deriv Cos } x) x)$
$\wedge$	$(\forall x \bullet (\textit{Cos Deriv } \sim(\textit{Sin } x)) x)$

Using `log` and `exp`, we can now easily derive some facts about roots, particularly square roots, that we will need to reason about the sine and cosine functions. The existence of roots could be proved by more elementary means using the intermediate value theorem, but now we have `log` and `exp`, it is much easier to use them instead. We also need some facts about even and odd numbers etc. to work with the terms of the power series for sine and cosine. We can then show using the power series that the definitions of sine and cosine are consistent, that they are uniquely specified by the definitions and so that they are equal to the expected power series expansions. (The series for cosine is not in a particularly tidy form — it is just what drops out from differentiating the series for sine term by term — but this could easily be remedied.) We then begin to develop their basic properties using the “axiomatic” approach following Mitchell [4]).

<i>positive_root_thm</i>	<i>sin_deriv_coeffs_thm</i>
<i>square_root_thm</i>	<i>sin_cos_consistency_thm</i>
<i>square_root_thm1</i>	<i>Sin_consistent</i>
<i>square_root_unique_thm</i>	<i>sin_def</i>
<i>square_square_root_mono_thm</i>	<i>sin_cos_cts_thm</i>
<i>sin_series_convergence_thm</i>	<i>cos_squared_plus_sin_squared_thm</i>
<i>even_odd_thm</i>	<i>sin_cos_unique_lemma1</i>
<i>mod_2_div_2_thm</i>	<i>sin_cos_unique_lemma2</i>
<i>mod_2_cases_thm</i>	<i>sin_cos_unique_thm</i>
<i>mod_2_0_R_N_exp_thm</i>	<i>sin_cos_power_series_thm</i>
<i>mod_2_1_R_N_exp_thm</i>	<i>sin_cos_plus_thm</i>
<i>power_series_even_thm</i>	<i>sin_cos_minus_thm</i>
<i>power_series_odd_thm</i>	

### 2.15.4 The Number $\pi$

Finally, we define Archimedes’ constant, the positive generator of the group of roots of the sine function:

HOL Constant

	<b><i>ArchimedesConstant</i></b> : $\mathbb{R}$
<hr/>	
	$\text{NR } 0 < \textit{ArchimedesConstant}$
$\wedge$	$\forall x \bullet \textit{Sin } x = \text{NR } 0$
	$\Leftrightarrow (\exists m \bullet x = \text{NR } m * \textit{ArchimedesConstant})$



|  $\vee$   $(\exists m \bullet x = \sim(\mathbb{NR} \ m * \text{ArchimedesConstant}))$

Archimedes' constant is, of course, almost always known by another name:

SML

| `declare_alias("π", «ArchimedesConstant»);`

We continue to develop the properties of sine and cosine via the axiomatic approach. Much of this part of the work amounts to a consistency proof for the above definition of  $\pi$ .

<i>sin_0_group_thm</i>	<i>sin_positive_estimate_thm</i>	<i>pi_consistency_thm</i>
<i>sin_eq_cos_sin_cos_twice_thm</i>	<i>cos_greater_root_2_thm</i>	<i>ArchimedesConstant_consistent</i>
<i>sum_squares_abs_bound_thm</i>	<i>cos_squared_eq_half_thm</i>	<i>π_def</i>
<i>sum_squares_abs_bound_thm1</i>	<i>sin_eq_cos_exists_thm</i>	<i>NR_plus_1_times_bound_thm</i>
<i>abs_sin_abs_cos_le_1_thm</i>	<i>sin_positive_zero_thm</i>	<i>NR_0_less_not_NR_times_π_thm</i>
<i>cos_positive_estimate_thm</i>	<i>ℝ_discrete_subgroup_thm</i>	

### 2.15.5 Periodicity of the Sine and Cosine Functions

We conclude the axiomatic development of the properties of sine and cosine by proving their periodic behaviour and finally, with a view to later geometric applications, show that any numbers  $x$  and  $y$  with  $x^2 + y^2 = 1$  are equal to  $\cos(z)$  and  $\sin(z)$  respectively for some, unique,  $z$  in the interval  $[0, 2\pi)$ .

<i>sin_not_eq_0_thm</i>	<i>sin_cos_plus_2_π_thm</i>
<i>sin_0_π_0_less_thm</i>	<i>sin_cos_2_π_thm</i>
<i>sin_cos_π_thm</i>	<i>sin_cos_plus_even_times_π_thm</i>
<i>sin_cos_π_over_2_thm</i>	<i>sin_cos_even_times_π_thm</i>
<i>sin_cos_plus_π_over_2_thm</i>	<i>sin_cos_period_thm</i>
<i>cos_not_eq_0_thm</i>	<i>sin_cos_onto_unit_circle_thm</i>
<i>cos_eq_0_thm</i>	<i>sin_cos_onto_unit_circle_thm1</i>
<i>sin_cos_plus_π_thm</i>	

## 2.16 L'Hôpital's Rule

In this section, we will state and prove (several formulations of) L'Hôpital's rule, which states that under appropriate conditions, if the ratio of two derivatives  $f'(y)/g'(y)$  has a limit then so does  $f(y)/g(y)$  and the two limits are equal. This result is most naturally stated and proved using one-sided limits rather than the two-sided limits we have been using to date. A two-sided reformulation is a trivial consequence of the one-sided one. Moreover, one wants to apply the theorems to limits at infinity (e.g., to evaluate the definite integrals that define the Laplace transform). In preparation for this, the next few subsections introduce the required new notions.

### 2.16.1 Right-hand Limits of Function Values

We will write  $(f+\#-\>c)x$  to indicate that  $f(y)$  tends to  $c$  as  $y$  tends to  $x$  from the right:

SML

```
| declare_infix(205, "+#->");
```

HOL Constant

```
| $+#-> : (ℝ → ℝ) → ℝ → ℝ → BOOL
```

---

```
| ∀f c a •
```

```
    (f +#-> c) a
```

```
| ⇔   ∀e•Nℝ 0 < e ⇒ ∃b•a < b ∧ ∀x•a < x ∧ x < b ⇒ Abs(f x - c) < e
```

As normal, we give a sequential convergence characterisation of this notion of limit and use it to calculate the right-hand limits of the usual algebraic combinations of functions. We also show that the right-hand limit at  $x$  of  $f$  is  $f(x)$  if  $f$  is continuous at  $x$  (*cts\_lim\_right\_thm*) and show that two-sided limits may be characterised using one-sided limits (*lim\_fun\_lim\_right\_thm*).

*lim\_right\_lim\_seq\_thm*

*lim\_fun\_lim\_right\_thm*

*const\_lim\_right\_thm*

*id\_lim\_right\_thm*

*plus\_lim\_right\_thm*

*times\_lim\_right\_thm*

*poly\_lim\_right\_thm*

*recip\_lim\_right\_thm*

*lim\_right\_unique\_thm*

*cts\_lim\_right\_thm*

## 2.16.2 Limits of Function Values at $+\infty$

We will write  $f-+\#>c$  to indicate that  $f(y)$  tends to  $c$  as  $y$  tends to  $+\infty$ :

SML

```
| declare_infix(205, "-+#>");
```

HOL Constant

```
| $-+#> : (ℝ → ℝ) → ℝ → BOOL
```

---

```
| ∀f c •
```

```
    f -+#> c
```

```
| ⇔   ∀e•Nℝ 0 < e ⇒ ∃b•∀x•b < x ⇒ Abs(f x - c) < e
```

As ever, we give a sequential convergence characterisation of this notion of limit and use it to calculate the limits at infinity of various functions. We also show how a limit at infinity can be expressed in terms of a limit at 0 and *vice versa*.

*lim\_infinity\_lim\_seq\_thm*

*lim\_infinity\_lim\_right\_thm*

*lim\_right\_lim\_infinity\_thm*

*const\_lim\_infinity\_thm*

*id\_lim\_infinity\_thm*

*plus\_lim\_infinity\_thm*

*times\_lim\_infinity\_thm*

*poly\_lim\_infinity\_thm*

*recip\_lim\_infinity\_thm*

*lim\_infinity\_unique\_thm*

(Note on the symbols: ‘#’ in the arrows means  $\infty$ ; ‘+ # -’ is then intended to suggest a passage along the real line to a point starting at  $+\infty$  and ‘- + #’ a passage in the opposite direction.)

### 2.16.3 Divergence to $\infty$ at $+\infty$

We will write  $f \dashv\vdash +\infty$  to indicate that  $f(y)$  diverges to  $+\infty$  as  $y$  tends to  $+\infty$ :

SML

```
| declare_postfix(200, "-+##>+##");
```

HOL Constant

```
| $-+##>+## : (ℝ → ℝ) → BOOL
```

---

```
|  $\forall f \bullet f \dashv\vdash +\infty \Leftrightarrow \forall x \bullet \exists b \bullet \forall y \bullet b < y \Rightarrow x < f y$ 
```

*div\_infinity\_pos\_thm*

*const\_plus\_div\_infinity\_thm*

*id\_div\_infinity\_thm*

*plus\_div\_infinity\_thm*

*const\_times\_div\_infinity\_thm*

*times\_div\_infinity\_thm*

*power\_div\_infinity\_thm*

*less\_div\_infinity\_thm*

*bounded\_deriv\_div\_infinity\_thm*

*exp\_div\_infinity\_thm*

*log\_div\_infinity\_thm*

*div\_infinity\_times\_recip\_thm*

*div\_infinity\_lim\_right\_thm*

*div\_infinity\_lim\_infinity\_thm*

### 2.16.4 Formulations of l'Hôpital's rule

We are now ready to formulate l'Hôpital's rule. In all the formulations this deals with two functions  $f$  and  $g$  such that the ratio  $f'(y)/g'(y)$  has a limit.

The first formulation is for right-hand limits under the following conditions: for some  $b > a$ ,  $f$  and  $g$  must be continuous in the half-closed interval  $[a, b)$ ;  $f$  and  $g$  must be differentiable in the open interval  $(a, b)$ ; the derivative  $g'(y)$  must not vanish in  $(a, b)$  while  $f(a) = f(b) = 0$ . The second formulation is as the first but stated for two-sided limits rather than right-hand limits and has similar conditions. The third formulation is for limits at  $+\infty$  under the following condition: for some  $a$ ,  $f$  and  $g$  must be differentiable in the half-closed interval  $(a, \infty)$ ; and the derivative  $g'(y)$  must not vanish in  $(a, \infty)$ .

A variant of Rolle's theorem is used to show that, under these conditions,  $g(y)$  does not vanish in a suitable open interval — this is rolled up in the Cauchy mean value theorem in the textbook proofs, but our 'symmetric' formulation of the latter theorem means that we need to establish this separately. A couple of lemmas about the continuity and derivatives of the function  $f(-x)$  are also handy in deducing the second formulation of the rule from the first.

*rolle\_thm1*

*l'hospital\_lim\_right\_thm*

*cts\_comp\_minus\_thm*

*deriv\_comp\_minus\_thm*

*l'hospital\_lim\_fun\_thm*

*l'hospital\_lim\_infinity\_thm*

### 2.16.5 Some Applications

We now put l'Hôpital's rule to work. The first application is a simple general result saying (in effect) that under suitable conditions limits of function values and derivatives commute.

Then we calculate some specific limits:  $x/\sin(x)$  as  $x$  tends to 0 (0);  $x^m/\exp(x)$  and  $\log(x)/x$  as  $x$  tends to  $+\infty$  (both 0);  $\log(x)/x$  as  $x$  tends to  $+\infty$  (1);  $\log(x)/(x-1)$  as  $x$  tends to 1 (and, hence,  $\log(1+x)/x$  as  $x$  tends to 0) (1); and, finally,  $(1+x)^{1/x} = \exp(\log(1+x)/x)$  as  $x$  tends to 0 ( $e = \exp(1)$ ), which leads to the famous result that  $e$  is the limit as  $n$  tends to infinity of  $(1+1/n)^n$ .

<i>cts_deriv_deriv_thm</i>	<i>lim_infinity_log_over_id_thm</i>
<i>lim_fun_id_over_sin_thm</i>	<i>lim_fun_log_over_id_minus_1_thm</i>
<i>lim_infinity_recip_exp_thm</i>	<i>lim_fun_log_1_plus_over_id_thm</i>
<i>lim_infinity_id_over_exp_thm</i>	<i>lim_fun_e_thm</i>
<i>lim_infinity_power_over_exp_thm</i>	<i>lim_seq_e_thm</i>

## 2.17 Further Functions

### 2.17.1 Inverse Functions

We have already dealt with the construction of one inverse function, namely  $\log$  as the inverse of  $\exp$ . The methods used were a little *ad hoc* and exploited some of the very nice particular properties of  $\exp$ .

We now develop some simple theory for dealing with the inverses of invertible continuous functions. We provide a selection of simple theorems that support some common cases. Later sections will apply the theorems to functions such as the square root function,  $\arcsin$  and  $\operatorname{arcsinh}$ . The cases we deal with here cover functions which are (*total\_inverse\_thm*) or which can be adapted to be (*closed\_half\_infinite\_inverse\_thm*, *closed\_interval\_inverse\_thm*) order-preserving bijections from the whole real line to itself. In all of these cases, the inverse is or can be chosen to be everywhere continuous. In applications this lets us specify an inverse of a continuous function defined over a closed interval, say, to be continuous at the end-points of its interval of definition.

<i><math>\mathbb{R}</math>_less_mono_↔_thm</i>	<i>closed_half_infinite_inverse_thm1</i>
<i><math>\mathbb{R}</math>_less_mono_↔_≤_thm</i>	<i>closed_half_infinite_inverse_thm</i>
<i><math>\mathbb{R}</math>_less_mono_one_one_thm</i>	<i>cond_cts_thm</i>
<i>total_inverse_cts_thm</i>	<i>cond_cts_thm1</i>
<i>total_inverse_thm</i>	<i>closed_interval_inverse_thm</i>

### 2.17.2 The Square Root Function

The square root function is an example of an inverse function defined on a closed half-infinite interval. Note that we can and do specify it to be continuous (in our two-sided sense of the word) throughout the interval of definition. This information is redundant in the interior of the interval, but not at the end-point. The extra detail can be very useful in applications (cf. the statement of numerous theorems such as the mean value theorem which require continuity but not differentiability at the end-points of an interval).

HOL Constant

**Sqrt** :  $\mathbb{R} \rightarrow \mathbb{R}$

$\forall x \bullet \mathbb{N}\mathbb{R} \ 0 \leq x \Rightarrow \mathbb{N}\mathbb{R} \ 0 \leq \text{Sqrt } x \wedge (\text{Sqrt } x) ^ 2 = x \wedge \text{Sqrt } \text{Cts } x$

We prove a selection of theorems about the square root function. Many of these are needed later for dealing with the derivatives of trigonometric functions etc.

*sqrt\_def*

*sqrt\_square\_thm*

*square\_sqrt\_thm*

*square\_times\_thm*

*sqrt\_times\_thm*

*square\_recip\_thm*

*sqrt\_pos\_thm*

*sqrt\_recip\_thm*

*sqrt\_exp\_log\_thm*

*sqrt\_deriv\_thm*

*id\_over\_sqrt\_thm*

*sqrt\_1\_minus\_sin\_squared\_thm*

*sqrt\_1\_minus\_cos\_squared\_thm*

### 2.17.3 More Trigonometric Functions

The standard definitions of the remaining trigonometric functions in terms of sin and cos are given in table 1.

HOL Constant

**Tan** :  $\mathbb{R} \rightarrow \mathbb{R}$

$\forall x \bullet \text{Tan } x = \text{Sin } x * \text{Cos } x ^{-1}$

HOL Constant

**Cotan** :  $\mathbb{R} \rightarrow \mathbb{R}$

$\forall x \bullet \text{Cotan } x = \text{Cos } x * \text{Sin } x ^{-1}$

HOL Constant

**Sec** :  $\mathbb{R} \rightarrow \mathbb{R}$

$\forall x \bullet \text{Sec } x = \text{Cos } x ^{-1}$

HOL Constant

**Cosec** :  $\mathbb{R} \rightarrow \mathbb{R}$

$\forall x \bullet \text{Cosec } x = \text{Sin } x ^{-1}$

Table 1: Further Trigonometric Functions

We provide theorems giving the derivatives of all of these functions. Apart from a little algebraic manipulation, the derivatives are all dealt with automatically by the tactic described in section 2.7.

*tan\_deriv\_thm*

*cotan\_deriv\_thm*

*sec\_deriv\_thm*

*cosec\_deriv\_thm*

We now define the inverse of the sine function, *arc sin*. It is an example of an inverse defined over a closed interval. The defining property below is somewhat redundant, but as the theorems about inverse functions that support the consistency proof deliver all this information readily, it is convenient to include it in the defining property. As with the square root function, we take the function to be continuous at the end-points of its interval of definition.

HOL Constant

**ArcSin** :  $\mathbb{R} \rightarrow \mathbb{R}$

---

$(\forall x \bullet \text{Abs } x \leq 1/2 * \pi \Rightarrow \text{ArcSin } (\text{Sin } x) = x)$   
 $\wedge (\forall x \bullet \text{Abs } x \leq \mathbb{NR } 1 \Rightarrow \text{Sin}(\text{ArcSin } x) = x \wedge \text{ArcSin } \text{Cts } x)$

After proving the consistency of the definition, we prove some lemmas which just provide the various parts of the definition in separate packaging and one or two simple algebraic properties of arc sin. These are then used to calculate its derivative.

*ArcSin\_consistent* *abs\_arc\_sin\_thm*  
*arc\_sin\_def* *arc\_sin\_sin\_thm*  
*sin\_arc\_sin\_thm* *cos\_arc\_sin\_thm*

*arc\_sin\_1\_minus\_1\_thm*  
*arc\_sin\_deriv\_thm*

For use in applications, we also define the inverse of the cosine function, arc cos and prove some elementary theorems about it.

HOL Constant

**ArcCos** :  $\mathbb{R} \rightarrow \mathbb{R}$

---

$(\forall x \bullet \mathbb{NR } 0 \leq x \wedge x \leq \pi \Rightarrow \text{ArcCos } (\text{Cos } x) = x)$   
 $\wedge (\forall x \bullet \text{Abs } x \leq \mathbb{NR } 1 \Rightarrow \text{Cos}(\text{ArcCos } x) = x \wedge \text{ArcCos } \text{Cts } x)$

*ArcCos\_consistent* *cos\_arc\_cos\_thm*  
*arc\_cos\_def* *abs\_arc\_cos\_thm*

*arc\_cos\_cos\_thm*  
*cos\_arc\_cos\_thm*

#### 2.17.4 Hyperbolic Trigonometric Functions

Table 2 presents the standard definitions of the six hyperbolic trigonometric functions in terms of exp.

As with the circular trigonometric functions, we provide theorems giving the derivatives of all these functions (together with one or two algebraic properties needed to calculate the derivatives). Again, apart from a little algebraic manipulation, the derivatives are all dealt with automatically by the tactic described in section 2.7.

*sinh\_deriv\_thm* *cosh\_non\_0\_thm*  
*cosh\_deriv\_thm* *sinh\_non\_0\_thm*  
*cosh\_0\_less\_thm* *cosh\_squared\_minus\_sinh\_squared\_thm*

<p>HOL Constant</p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><b><i>Sinh</i></b> : <math>\mathbb{R} \rightarrow \mathbb{R}</math></p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><math>\forall x \bullet \text{Sinh } x = 1/2 * (\text{Exp } x - \text{Exp } (\sim x))</math></p>	<p>HOL Constant</p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><b><i>Cosh</i></b> : <math>\mathbb{R} \rightarrow \mathbb{R}</math></p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><math>\forall x \bullet \text{Cosh } x = 1/2 * (\text{Exp } x + \text{Exp } (\sim x))</math></p>
<p>HOL Constant</p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><b><i>Tanh</i></b> : <math>\mathbb{R} \rightarrow \mathbb{R}</math></p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><math>\forall x \bullet \text{Tanh } x = \text{Sinh } x * \text{Cosh } x^{-1}</math></p>	<p>HOL Constant</p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><b><i>Cotanh</i></b> : <math>\mathbb{R} \rightarrow \mathbb{R}</math></p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><math>\forall x \bullet \text{Cotanh } x = \text{Cosh } x * \text{Sinh } x^{-1}</math></p>
<p>HOL Constant</p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><b><i>Sech</i></b> : <math>\mathbb{R} \rightarrow \mathbb{R}</math></p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><math>\forall x \bullet \text{Sech } x = \text{Cosh } x^{-1}</math></p>	<p>HOL Constant</p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><b><i>Cosech</i></b> : <math>\mathbb{R} \rightarrow \mathbb{R}</math></p> <hr style="width: 100%; border: 0.5px solid black;"/> <p><math>\forall x \bullet \text{Cosech } x = \text{Sinh } x^{-1}</math></p>

Table 2: The Hyperbolic Trigonometric Functions

*tanh\_deriv\_thm*  
*cotanh\_deriv\_thm*  
*sech\_deriv\_thm*

We also introduce the inverse, *arc sinh*, of *sinh*. It is an inverse function of the simplest type, since the domain and range of *sinh* are the whole real line. As with the square root and *arc sin* functions, the method of proving the definition consistent gives us the continuity of the function for free.

HOL Constant

---

***ArcSinh*** :  $\mathbb{R} \rightarrow \mathbb{R}$

---

$\forall x \bullet \text{ArcSinh } (\text{Sinh } x) = x \wedge \text{Sinh}(\text{ArcSinh } x) = x \wedge \text{ArcSinh } \text{Cts } x$

After proving the consistency of the definition, we derive a couple of algebraic properties and use them to calculate the derivative.

*ArcSinh\_consistent*                      *sqrt\_1\_plus\_sinh\_squared\_thm*    *arc\_sinh\_deriv\_thm*  
*arc\_sinh\_def*                                *cosh\_arc\_sinh\_thm*

## 2.18 Integration

The Henstock-Kurzweil approach to the integral calculus is very well-suited to our needs. In addition to the advantages listed by John Harrison [2], I note that: (i) the Henstock-Kurzweil gauge integral is not restricted to functions with a bounded range and can naturally be formulated so that integrals over bounded and unbounded intervals fall under a single definition; (ii) the fundamental theorem of the calculus for the gauge integral can accommodate an antiderivative that “fails” at some points of the interval of integration.

Both these points are illustrated by the example  $\int_{x=-1}^{+1} dx/\sqrt{1-x^2}$ . The fundamental theorem for the gauge integral shows this to be equal to  $[\text{arc sin}(x)]_{-1}^{+1} = \pi$  even though at the points  $-1$  and  $+1$  the antiderivative  $\text{arc sin}(x)$  is not differentiable and the integrand  $1/\sqrt{1-x^2}$  tends to  $+\infty$ . This integral would be much more complicated to deal with under the Riemann theory.

### 2.18.1 The Henstock-Kurzweil Gauge Integral Defined

We now define the Henstock-Kurzweil gauge integral, see, e.g., the book by Kurtz and Swartz [3]. Our approach is to take integrals over the whole line as fundamental and to define integrals over subsets such as intervals as a derived notion.

The Henstock-Kurzweil theory deals with estimates to the putative value of an integral defined over what we call, following Kurtz and Swartz, *tagged partitions* (also called “tagged divisions” in the literature). A tagged partition of an interval comprises a (finite) contiguous sequence of closed intervals,  $[I_0, I_1], [I_1, I_2], \dots, [I_n, I_{n+1}]$  together with sample points,  $t_0 \in [I_0, I_1], t_1 \in [I_1, I_2], \dots, t_n \in [I_n, I_{n+1}]$ . We represent the sequence of intervals by the real-valued sequence giving the end-points (and for simplicity, we require each interval to have a non-empty interior). This leads to the following definition:

HOL Constant

$$\begin{array}{|l} \mathbf{TaggedPartition} : ((\mathbb{N} \rightarrow \mathbb{R}) \times (\mathbb{N} \rightarrow \mathbb{R}) \times \mathbb{N}) \text{ SET} \\ \hline \forall t \ I \ n \bullet \quad (t, I, n) \in \mathbf{TaggedPartition} \\ \Leftrightarrow \quad (\forall m \bullet m < n \Rightarrow I \ m < I \ (m+1)) \\ \wedge \quad (\forall m \bullet m < n \Rightarrow t \ m \in \mathbf{ClosedInterval} \ (I \ m) \ (I \ (m+1))) \end{array}$$

The *Riemann sum* associated with a real-valued function and a tagged partition is then the estimate given by taking the sums of the values at the sample points each weighted by the length of the corresponding sample interval. Recalling that *Series* is the function that maps a sequence,  $s_m$ , to its sequence of partial sums  $\sum_{n=0}^{m-1} s_n$ , we capture this as follows:

HOL Constant

$$\begin{array}{|l} \mathbf{RiemannSum} : (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow ((\mathbb{N} \rightarrow \mathbb{R}) \times (\mathbb{N} \rightarrow \mathbb{R}) \times \mathbb{N}) \rightarrow \mathbb{R} \\ \hline \forall f \ t \ I \ n \bullet \quad \mathbf{RiemannSum} \ f \ (t, I, n) = \mathbf{Sigma} \ (\lambda m \bullet f(t \ m) * (I(m+1) - I \ m)) \ n \end{array}$$

The Riemann integral of a function  $f$  over an interval  $[a, b]$  is given by a limit as  $\delta > 0$  tends to zero of Riemann sums of  $f$  formed over tagged partitions that cover the interval and whose mesh (maximum length of interval) is bounded by  $\delta$ . The Henstock-Kurzweil theory generalises this limiting process to take into account local behaviour of  $f$ . A local bound on the interval lengths in a tagged partition is given by what is called a *gauge*, which we define to be a function that maps each real number  $x$  to one of its open neighbourhoods.

HOL Constant

$$\begin{array}{|l} \mathbf{Gauge} : (\mathbb{R} \rightarrow \mathbb{R} \text{ SET}) \text{ SET} \\ \hline \forall G \bullet G \in \mathbf{Gauge} \Leftrightarrow \forall x \bullet G \ x \in \mathbf{Open}_R \ \wedge \ x \in G \ x \end{array}$$

Note: the theory is not affected if gauges are restricted to open intervals rather than arbitrary open neighbourhoods or even intervals that are symmetric about the point in question (in which



case a gauge reduces, essentially, to a positive function giving the radii of the symmetric intervals). However, it seems best to have as much freedom as one can in designing gauges (since that is the crux of many of the proofs).

A tagged partition is *fine* with respect to a gauge if each of its interval is contained in the open set that the gauge associates with the corresponding tag.

SML

```
| declare_postfix(330, "Fine");
```

HOL Constant

```
| $Fine : (ℝ → ℝ SET) → ((ℕ → ℝ) × (ℕ → ℝ) × ℕ) SET
```

---

```
| ∀t I n G•
```

```
|     (t, I, n) ∈ G Fine
```

```
| ⇔     (∀m• m < n ⇒ ClosedInterval (I m) (I (m+1)) ⊆ G(t m))
```

We now define the gauge integral of a function over the whole real line. A function  $f$  has integral  $c$  iff. for any positive  $e$ , there is a gauge  $G$  and real values  $a < b$ , such that any Riemann sum of  $f$  over a  $G$ -fine tagged partition that covers the interval  $[a, b]$  is within  $e$  of  $c$ . When this is the case, we write  $f \text{ Int}_R c$ , which corresponds to the informal statement that the integral  $\int_{x=-\infty}^{+\infty} f(x)dx$  exists and is equal to  $c$ .

SML

```
| declare_infix(200, "Int_R");
```

HOL Constant

```
| $Int_R : (ℝ → ℝ) → ℝ → BOOL
```

---

```
| ∀f c•
```

```
|     (f Int_R c)
```

```
| ⇔     ∀e• NR 0 < e ⇒
```

```
|         ∃G a b•     G ∈ Gauge ∧ a < b
```

```
|         ∧ ∀t I n•   (t, I, n) ∈ TaggedPartition
```

```
|             ∧     I 0 < a ∧ b < I n ∧ (t, I, n) ∈ G Fine
```

```
|             ⇒     Abs(RiemannSum f (t, I, n) - c) < e
```

(Those familiar with the theory formulated over a bounded interval  $[a, b]$ , should think of our partitions as modelling a partition of the extended real line  $[-\infty, +\infty]$  with the function  $f$  extended to have the value 0 at  $\pm\infty$  and with the sample points in the half-infinite intervals restricted to lie at  $\pm\infty$ , so that those intervals make no contribution to the Riemann sums. We will prove (*bounded\_int\_thm*) that our definition agrees with the other formulation for integrals over bounded intervals.)

To define integrals over bounded intervals, we use the notion of characteristic function, which we can conveniently define polymorphically rather than just for sets of real numbers.

HOL Constant

```
| CharFun : 'a SET → 'a → ℝ
```

---

```
| ∀A x• CharFun A x = if x ∈ A then NR 1 else NR 0
```

SML

```
| declare_alias("χ", ⊔ CharFun ⊔);
```

We now define the gauge integral of  $f$  over a set  $A$  to be the integral over the whole real line of the function defined to agree with  $f$  on  $A$  and to be identically zero off  $A$ .  $A$  will, of course, often be an interval, in which case the formal notation  $(f \text{ Int } c)$  (*ClosedInterval a b*) corresponds to the informal statement that the integral  $\int_{x=a}^b f(x)dx$  exists and is equal to  $c$ .

SML

```
| declare_infix(200, "Int");
```

HOL Constant

```
| $Int : (ℝ → ℝ) → ℝ → ℝ SET → BOOL
```

```
| ⊔ f c A • (f Int c) A ⇔ (λx • χ A x * f x) Int_R c
```

### 2.18.2 Properties of the Integral

The development of the theory begins with some generalities about gauges. One says that one gauge,  $G1$ , refines another,  $G2$ , if  $G1(x) \subseteq G2(x)$ , for every  $x$  (which means that  $G1$ -finess implies  $G2$ -finess). Any 2 gauges have a greatest common refinement given by intersection of their values. New gauges can be formed from old by composing them with a reflection, a translation or a dilation.

*gauge\_∩\_thm*

*gauge\_o\_minus\_thm*

*gauge\_o\_plus\_thm*

*gauge\_o\_times\_thm*

*gauge\_refinement\_thm*

*gauge\_refine\_3\_thm*

*fine\_refinement\_thm*

What I call the *chosen tag theorem* is very useful in designing gauges to achieve special effects: it says that for any real number  $a$ , there is a gauge  $G$  such that any tagged partition which is  $G$ -fine has  $a$  as the sample point in any of its intervals that contains  $a$ . We also have an analogous theorem for a finite set of chosen tags. We then have a theorem which asserts the existence of the “uniform” gauges that characterise the Riemann integral.

Finally in this block we have Cousin’s lemma, which says that, if a gauge  $G$  and  $a \leq b$  are given, then there exists a  $G$ -fine tagged partition beginning at  $a$  and ending at  $b$ . The proof is very neat via Bolzano’s principle of bisection.

*chosen\_tag\_thm*

*chosen\_tags\_thm*

*riemann\_gauge\_thm*

*cousin\_lemma*

We need to develop some elementary theory to deal with Riemann sums, tagged partitions and so on. This is given in the following block of lemmas, showing, for example, that the Riemann sum operation is linear in its function argument. The last of these lemmas captures the fact that a given point can occur as the sample point in at most two of the intervals in a tagged partition (and if it does occur in two, then the two are consecutive).

<i>riemann_sum_induction_thm</i>	<i>partition_reverse_clauses</i>	<i>tag_lower_bound_thm</i>
<i>series_induction_thm1</i>	<i>tagged_partition_append_thm</i>	<i>subpartition_thm</i>
<i>ℝ_abs_series_thm</i>	<i>fine_append_thm</i>	<i>subpartition_fine_thm</i>
<i>riemann_sum_induction_thm1</i>	<i>riemann_sum_0_thm</i>	<i>riemann_sum_local_thm1</i>
<i>riemann_sum_plus_thm</i>	<i>riemann_sum_append_thm</i>	<i>riemann_sum_0_≤_thm</i>
<i>riemann_sum_const_times_thm</i>	<i>riemann_sum_¬_0_thm</i>	<i>tag_unique_thm</i>
<i>riemann_sum_minus_thm</i>	<i>partition_mono_thm</i>	<i>partition_cover_thm</i>
<i>riemann_sum_local_thm</i>	<i>tag_mono_thm</i>	<i>partition_cover_thm1</i>
<i>riemann_sum_o_minus_thm</i>	<i>tag_upper_bound_thm</i>	<i>riemann_sum_χ_singleton_thm</i>
<i>riemann_sum_o_times_thm</i>		

Now we can start proving basic facts about the gauge integral. Our first few lemmas show that it is a linear operator . We then prove that integrals are unique when they exist. This follows from Cousin's lemma and linearity

<i>int_ℝ_plus_thm</i>	<i>int_ℝ_0_thm</i>	<i>int_ℝ_diff_0_thm</i>
<i>int_ℝ_minus_thm</i>	<i>int_ℝ_const_times_thm</i>	<i>int_ℝ_unique_thm</i>

There are then some results mainly about functions with integral 0, but beginning with the fact that integration is weakly order-preserving (taking the pointwise ordering of functions). If a non-negative function,  $f$ , is dominated by a function,  $g$ , and if  $g$  is integrable with integral 0, then so is  $f$ . The characteristic functions of singleton sets are integrable with integral 0, and, hence, so are all functions with finite support.

<i>int_ℝ_0_≤_thm</i>	<i>int_ℝ_χ_singleton_lemma</i>	<i>int_ℝ_singleton_support_thm</i>
<i>int_ℝ_≤_thm</i>	<i>int_ℝ_χ_singleton_thm</i>	<i>int_ℝ_finite_support_thm</i>
<i>int_ℝ_0_dominated_thm</i>		

The next few theorems are about sets of measure 0, i.e., sets  $A$  such that the characteristic function  $\chi_A$  is integrable with integral 0. The set of sets of measure 0 is downwards closed and forms a join semilattice.

<i>int_ℝ_χ_0_⊆_thm</i>	<i>χ_∪_∩_thm</i>
<i>int_ℝ_χ_0_∩_thm</i>	<i>int_ℝ_χ_0_∪_thm</i>

The first few lemmas in the next block show (in various useful guises) that the integral is not affected by the value of the function at a selected point. So for example, integration over a closed interval,  $[a, b]$  is equivalent to integration over the corresponding open interval,  $(a, b)$  We then investigate how the integral is transformed under linear change of variables. The remaining results here show that our definition of the integral over a closed interval agrees with the more normal one (defined in terms of tagged partitions that precisely cover the interval).

<i>int_half_infinite_interval_thm</i>	<i>int_ℝ_support_bounded_below_thm</i>
<i>chosen_value_thm</i>	<i>int_support_bounded_below_lemma</i>
<i>int_interval_thm</i>	<i>int_bounded_below_thm</i>
<i>int_ℝ_o_minus_thm</i>	<i>int_ℝ_bounded_support_thm</i>
<i>int_ℝ_o_plus_thm</i>	<i>bounded_int_thm</i>
<i>int_ℝ_o_times_thm</i>	<i>bounded_int_local_thm</i>

### 2.18.3 The Fundamental Theorem of the Calculus

We move on to prove the fundamental theorem of the calculus saying that if  $sf$  has derivative  $f$  in some open interval  $(a, b)$  and if  $sf$  is continuous at  $a$  and  $b$  then  $f$  is integrable on  $[a, b]$  with integral equal to  $sf(b) - sf(a)$  (this is *int\_deriv\_thm* below, the other variants corresponding to the common cases where  $sf$  is differentiable at one or both of the end-points of the intervals).

The key lemma in the result is the so-called *straddle theorem*. We use this to prove the result in the case where  $sf$  is required to be differentiable at  $a$ , but not at  $b$ , the other cases being easy consequences of that case. In outline, the proof uses the straddle lemma to show that the fundamental theorem holds over  $[a, x]$  for all  $x$  in  $(a, b)$  and then the continuity of  $sf$  at  $b$  to show that the integral over  $[a, b]$  exists and is equal to the limit of the integrals over  $[a, x]$  as  $x$  tends to  $b$  which gives the desired result. In fact, the fundamental theorem of the calculus can be made to accommodate countably many points of failure in the antiderivative, but the argument is a little harder and we have not yet needed the full strength of the result.

<i>straddle_thm</i>	<i>int_deriv_thm2</i>	<i>int_deriv_thm</i>
<i>straddle_gauge_thm</i>	<i>int_deriv_thm3</i>	<i>int_deriv_thm1</i>

Armed with the fundamental theorem, we calculate the integrals of some example integrands: the characteristic function of an interval; the example  $1/\sqrt{1-x^2}$  mentioned at the beginning of this section; and the reciprocal function.

<i>int_ℝ_χ_interval_thm</i>	<i>int_example_thm</i>	<i>int_recip_thm</i>
-----------------------------	------------------------	----------------------

We put the last of these theorems to work to show that the harmonic series  $s_i = 1/m$  is divergent: the theorem says that if  $0 < a < b$ , then  $\int_a^b (1/x) dx = \log(b) - \log(a)$ . Since  $1/x \leq 1/a$  throughout this interval we have that  $\log(b) - \log(a) \leq \int_a^b (1/a) dx = (b-a)/a$ . Thus for positive integer  $m$ , taking  $a = m$ ,  $b = m+1$ , we have that  $\log(m+1) - \log(m) \leq 1/m$ . Adding these inequalities for  $1 \leq m \leq n$ , gives that  $\log(n) \leq \sum_{m=1}^n 1/m$  and so the latter sequence of partial sums must diverge as  $n$  tends to  $\infty$ . The following block of theorems implement this argument.

<i>log_minus_log_estimate_thm</i>	<i>harmonic_series_estimate_thm</i>	<i>harmonic_series_divergent_thm</i>
-----------------------------------	-------------------------------------	--------------------------------------

## 2.18.4 Application: Areas of Plane Sets

The standard Lebesgue measure on the real line can be defined using the gauge integral, the measure of a set,  $A$ , being the integral,  $\int_{x=-\infty}^{\infty} \chi_A(x) dx$ , of its characteristic function. Analogously, we may define the *area* of a plane set  $A$ , to be given by the following double integral<sup>1</sup>:

$$\int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \chi_A(x, y) dy dx = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \chi_{\{y | (x, y) \in A\}}(y) dy dx$$

In the formal treatment, we will write ‘ $A$  Area  $c$ ’ to indicate that the plane set  $A$  has area  $c$ , as given by the above double integral, if it exists. The following definition captures this using the “curried” form of the integrand given on the right-hand side of the equation above. The function  $sf$  in the definition gives the inner integral.

SML

```
| declare_infix(200, "Area");
```

HOL Constant

```
| $Area : (ℝ × ℝ) SET → ℝ → BOOL
```

---

```
| ∀A c • (A Area c) ⇔ ∃sf • sf Int_R c ∧ (∀x • χ {y | (x, y) ∈ A} Int_R sf x)
```

We first prove some generalities which are useful in calculations and which help to verify that we have given the definition correctly. Firstly, and very importantly, the area of a set is unique if it is defined; areas are invariant under translations; under a dilation with scale factors  $d \neq 0$  along the  $x$ -axis and  $e \neq 0$  along the  $y$ -axis, areas are multiplied by  $|d| \times |e|$ ; the area of the empty set is 0; if  $A$  and  $B$  have areas, then  $A \cup B$  has an area iff.  $A \cap B$  does and in that case the four areas are related by the inclusion/exclusion principle (the two directions of the bi-implication are given as separate theorems below).

*area\_unique\_thm*

*area\_dilate\_thm1*

*area\_∪\_thm*

*area\_translate\_thm*

*area\_empty\_thm*

*area\_∩\_thm*

*area\_dilate\_thm*

We now calculate some specific areas. The area of a rectangle of width  $w$  and height  $h$  is  $w \times h$ . The proof implements the following evaluation of the double integral giving the area:

$$\int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \chi_{\{y | x \in [0, w] \wedge y \in [0, h]\}}(y) dy dx = \int_{x=0}^w \int_{y=0}^h 1 dy dx = \int_{x=0}^w h dy dx = w \times h$$

*area\_rectangle\_thm*

We next calculate the area bounded by a circle of radius  $r$ , i.e., the area of the set of points  $(x, y)$  such that  $\sqrt{x^2 + y^2} \leq r$ . We proceed in a sequence of lemmas implementing the following argument: first consider the case  $r = 1$ , so that the set we are interested in is bounded by the graphs of the

---

<sup>1</sup>The resulting notion is slightly less general than the standard Lebesgue measure on the plane, but bounded Lebesgue measurable sets which fail to have an area in this sense are quite exotic. In any case, the definition matches the standard and time-honoured way of calculating areas in practice.

functions  $\sqrt{1-x^2}$  and  $-\sqrt{1-x^2}$  between the end-points  $x = -1$  and  $x = 1$ . The inner integral in the double integral giving the area is therefore equal to  $2\sqrt{1-x^2}$ :

$$\int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} \chi_{\{|y|\sqrt{x^2+y^2} \leq 1\}}(y) dy dx = \int_{x=-1}^{+1} 2\sqrt{1-x^2} dx$$

The substitution  $x = \sin \theta$  suggests the antiderivative  $x\sqrt{1-x^2} + \arcsin(x)$  for the integrand  $2\sqrt{1-x^2}$ . Subject to a little algebraic simplification, this may be verified mechanically for  $x$  in the open interval  $(-1, 1)$ . As the antiderivative is continuous throughout  $[-1, 1]$ , we can apply the fundamental theorem of the calculus<sup>2</sup> to conclude that

$$\int_{x=-1}^{+1} 2\sqrt{1-x^2} dx = \left[ x\sqrt{1-x^2} + \arcsin(x) \right]_{x=-1}^{+1} = (0 + \pi/2) - (0 - \pi/2) = \pi.$$

That the area bounded by a circle of radius  $r$  is  $\pi r^2$  now follows from what we have proved about the behaviour of areas under dilations (given that the set in question is the image of the set bounded by the unit circle under a dilation with scale factor  $r$  in both directions).

*area\_circle\_lemma1*

*area\_circle\_lemma4*

*circle\_dilate\_thm*

*area\_circle\_lemma2*

*area\_circle\_int\_thm*

*area\_circle\_thm*

*area\_circle\_lemma3*

*area\_unit\_circle\_thm*

Using this notion of area, we can calculate probabilities of events parametrised by pairs of real numbers: If  $S \subseteq \mathbb{R} \times \mathbb{R}$  is a sample space with area  $s \neq 0$  say, and  $X \subseteq S$  has area  $x$ , then the probability that an event in  $S$  belongs to  $X$  is  $x/s$ . In this way, we may state and prove the Buffon needle theorem: this says that if a needle of unit length is dropped at random onto the plane, then the probability that the needle crosses one of the lines  $y = m$  with  $m \in \mathbb{Z}$  is  $2/\pi$ .

We take as the parameters *(i)*, the angle,  $\theta$ , between the needle and the horizontal, and, *(ii)*, the distance,  $d$ , between the uppermost end of the needle and the line  $y = m$  immediately below it. So  $S$  is the set  $[0, \pi] \times [0, 1]$  and  $X$  comprises those  $(\theta, d) \in S$  such that the line segment between  $(0, d)$  and  $(\cos(\theta), d - \sin(\theta))$  crosses the  $x$ -axis.  $X$  and  $S$  have areas  $2$  and  $\pi$  respectively which gives the stated probability of  $2/\pi$ . The proofs of the theorems named below implement this argument with the calculation of the area of  $X$  as a separate lemma.

*buffon\_needle\_lemma*

*buffon\_needle\_thm*

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<sup>2</sup>The antiderivative can be adjusted to be valid at the end-points of the integral too, but proving that involves additional effort and the proof is not routine. The gauge integral comes into its own here in saving us work.

# A THE THEORY analysis

## A.1 Parents

*fin\_set*  $\mathbb{R}$

## A.2 Constants

<b><i>PolyFunc</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \mathbb{P}$
<b><i>PolyEval</i></b>	$\mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$
<b><i>PlusCoeffs</i></b>	$\mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST}$
<b><i>TimesCoeffs</i></b>	$\mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST}$
<b><i>\$To</i></b>	$(\mathbb{N} \rightarrow 'a) \rightarrow \mathbb{N} \rightarrow 'a \text{ LIST}$
<b><i>Roots</i></b>	$('a \rightarrow \mathbb{R}) \rightarrow 'a \mathbb{P}$
<b><i>\$-&gt;</i></b>	$(\mathbb{N} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b><i>ClosedInterval</i></b>	$\mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R} \mathbb{P}$
<b><i>OpenInterval</i></b>	$\mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R} \mathbb{P}$
<b><i>\$Cts</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b><i>\$Deriv</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b><i>DerivCoeffs</i></b>	$\mathbb{R} \text{ LIST} \rightarrow \mathbb{R} \text{ LIST}$
<b><i>Open<sub>R</sub></i></b>	$\mathbb{R} \mathbb{P} \mathbb{P}$
<b><i>Closed<sub>R</sub></i></b>	$\mathbb{R} \mathbb{P} \mathbb{P}$
<b><i>Compact<sub>R</sub></i></b>	$\mathbb{R} \mathbb{P} \mathbb{P}$
<b><i>\$--&gt;</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b><i>\$---&gt;</i></b>	$(\mathbb{N} \rightarrow \mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \mathbb{P} \rightarrow \text{BOOL}$
<b><i>Sigma</i></b>	$(\mathbb{N} \rightarrow \mathbb{R}) \rightarrow \mathbb{N} \rightarrow \mathbb{R}$
<b><i>PowerSeries</i></b>	$(\mathbb{N} \rightarrow \mathbb{R}) \rightarrow \mathbb{N} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$
<b><i>\$!</i></b>	$\mathbb{N} \rightarrow \mathbb{N}$
<b><i>Exp</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Log</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Cos</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Sin</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>ArchimedesConstant</i></b>	$\mathbb{R}$
<b><i>\$+#-&gt;</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b><i>\$-+#&gt;</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b><i>\$-+#&gt;+#</i></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \text{BOOL}$
<b><i>Sqrt</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Tan</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Cotan</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Sec</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Cosec</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>ArcSin</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>ArcCos</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Sinh</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Cosh</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Tanh</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Cotanh</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Sech</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b><i>Cosech</i></b>	$\mathbb{R} \rightarrow \mathbb{R}$

<b>ArcSinh</b>	$\mathbb{R} \rightarrow \mathbb{R}$
<b>TaggedPartition</b>	$(\mathbb{N} \rightarrow \mathbb{R}) \leftrightarrow ((\mathbb{N} \rightarrow \mathbb{R}) \times \mathbb{N})$
<b>RiemannSum</b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow (\mathbb{N} \rightarrow \mathbb{R}) \times (\mathbb{N} \rightarrow \mathbb{R}) \times \mathbb{N} \rightarrow \mathbb{R}$
<b>Gauge</b>	$(\mathbb{R} \rightarrow \mathbb{R} \mathbb{P}) \mathbb{P}$
<b>\$Fine</b>	$(\mathbb{R} \rightarrow \mathbb{R} \mathbb{P}) \rightarrow (\mathbb{N} \rightarrow \mathbb{R}) \leftrightarrow ((\mathbb{N} \rightarrow \mathbb{R}) \times \mathbb{N})$
<b>\$Int<sub>R</sub></b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \text{BOOL}$
<b>CharFun</b>	$'a \mathbb{P} \rightarrow 'a \rightarrow \mathbb{R}$
<b>\$Int</b>	$(\mathbb{R} \rightarrow \mathbb{R}) \rightarrow \mathbb{R} \rightarrow \mathbb{R} \mathbb{P} \rightarrow \text{BOOL}$
<b>\$Area</b>	$\mathbb{R} \leftrightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \text{BOOL}$

### A.3 Aliases

$\pi$	<i>ArchimedesConstant</i> : $\mathbb{R}$
$\chi$	<i>CharFun</i> : $'a \mathbb{P} \rightarrow 'a \rightarrow \mathbb{R}$

### A.4 Fixity

*Right Infix 200:*

**Area Cts**    **Deriv Int**    **Int<sub>R</sub>**     $\rightarrow$

*Right Infix 205:*

$\rightarrow$      $\rightarrow$      $\rightarrow$      $\rightarrow$

*Right Infix 310:*

**To**

*Postfix 200:*     $\rightarrow$      $\rightarrow$

*Postfix 330:*    **Fine !**

### A.5 Definitions

<b>PolyFunc</b>	$\vdash \text{PolyFunc}$ $= \bigcap$ $\{A$ $    (\forall c \bullet (\lambda x \bullet c) \in A)$ $  \wedge (\lambda x \bullet x) \in A$ $  \wedge (\forall f g$ $    \bullet f \in A \wedge g \in A \Rightarrow (\lambda x \bullet f x + g x) \in A)$ $  \wedge (\forall f g$ $    \bullet f \in A \wedge g \in A \Rightarrow (\lambda x \bullet f x * g x) \in A)\}$
<b>PolyEval</b>	$\vdash (\forall x \bullet \text{PolyEval } [] x = 0.)$ $\wedge (\forall c l x$ $  \bullet \text{PolyEval } (\text{Cons } c l) x = c + x * \text{PolyEval } l x)$
<b>PlusCoeffs</b>	$\vdash \text{ConstSpec}$ $(\lambda \text{ PlusCoeffs}'$ $  \bullet (\forall l \bullet \text{PlusCoeffs}' [] l = l)$ $  \wedge (\forall l \bullet \text{PlusCoeffs}' l [] = l)$ $  \wedge (\forall c1 l1 c2 l2$ $    \bullet \text{PlusCoeffs}' (\text{Cons } c1 l1) (\text{Cons } c2 l2)$ $      = \text{Cons } (c1 + c2) (\text{PlusCoeffs}' l1 l2)))$ $\text{PlusCoeffs}$
<b>TimesCoeffs</b>	$\vdash (\forall l \bullet \text{TimesCoeffs } [] l = [])$ $\wedge (\forall c l1 l2$



- $TimesCoeffs (Cons\ c\ l1)\ l2$   
 $= PlusCoeffs$   
 $(Cons\ 0.\ (TimesCoeffs\ l1\ l2))$   
 $(Map\ (\lambda\ x\bullet\ c\ *\ x)\ l2))$

**To**  $\vdash (\forall f\bullet f\ To\ 0 = [])$   
 $\wedge (\forall f\ n\bullet f\ To\ (n + 1) = f\ To\ n \wedge [f\ n])$

**Roots**  $\vdash \forall f\bullet Roots\ f = \{x|f\ x = 0.\}$

$\rightarrow$   $\vdash \forall s\ x$ 

- $s \rightarrow x$   
 $\Leftrightarrow (\forall e$   
  - $0. < e \Rightarrow (\exists n\bullet \forall m\bullet n \leq m \Rightarrow Abs\ (s\ m - x) < e))$

**ClosedInterval**  $\vdash \forall x\ y\bullet ClosedInterval\ x\ y = \{t|x \leq t \wedge t \leq y\}$

**OpenInterval**  $\vdash \forall x\ y\bullet OpenInterval\ x\ y = \{t|x < t \wedge t < y\}$

**Cts**  $\vdash \forall f\ x$ 

- $f\ Cts\ x$   
 $\Leftrightarrow (\forall e$   
  - $0. < e$   
 $\Rightarrow (\exists d$   
    - $0. < d$   
 $\wedge (\forall y$   
      - $Abs\ (y - x) < d$   
 $\Rightarrow Abs\ (f\ y - f\ x) < e))$

**Deriv**  $\vdash \forall f\ c\ x$ 

- $(f\ Deriv\ c)\ x$   
 $\Leftrightarrow (\forall e$   
  - $0. < e$   
 $\Rightarrow (\exists d$   
    - $0. < d$   
 $\wedge (\forall y$   
      - $Abs\ (y - x) < d \wedge \neg\ y = x$   
 $\Rightarrow Abs\ ((f\ y - f\ x) / (y - x) - c) < e))$

**DerivCoeffs**  $\vdash DerivCoeffs\ [] = []$   
 $\wedge (\forall c\ l$   

- $DerivCoeffs\ (Cons\ c\ l)$   
 $= PlusCoeffs\ l\ (Cons\ 0.\ (DerivCoeffs\ l))$

**Open<sub>R</sub>**  $\vdash Open_R$   
 $= \{A$   
 $|\forall t$   

- $t \in A$   
 $\Rightarrow (\exists x\ y$   
  - $t \in OpenInterval\ x\ y$   
 $\wedge OpenInterval\ x\ y \subseteq A)\}$

**Closed<sub>R</sub>**  $\vdash Closed_R = \{A|\sim A \in Open_R\}$

**Compact<sub>R</sub>**  $\vdash Compact_R$   
 $= \{A$   
 $|\forall V$   

- $V \subseteq Open_R \wedge A \subseteq \bigcup V$   
 $\Rightarrow (\exists W\bullet W \subseteq V \wedge W \in Finite \wedge A \subseteq \bigcup W)\}$

$\dashrightarrow$   $\vdash \forall f\ c\ x$

$\bullet (f \dashrightarrow c) x$   
 $\Leftrightarrow (\forall e$   
 $\bullet 0. < e$   
 $\Rightarrow (\exists d$   
 $\bullet 0. < d$   
 $\wedge (\forall y$   
 $\bullet Abs (y - x) < d \wedge \neg y = x$   
 $\Rightarrow Abs (f y - c) < e)))$

$\dashrightarrow$   $\vdash \forall u h X$   
 $\bullet (u \dashrightarrow h) X$   
 $\Leftrightarrow (\forall e$   
 $\bullet 0. < e$   
 $\Rightarrow (\exists n$   
 $\bullet \forall m y$   
 $\bullet n \leq m \wedge y \in X \Rightarrow Abs (u m y - h y) < e))$

**Sigma**  $\vdash (\forall s \bullet Sigma s 0 = 0.)$   
 $\wedge (\forall s n \bullet Sigma s (n + 1) = Sigma s n + s n)$

**PowerSeries**  $\vdash \forall s n \bullet PowerSeries s n = PolyEval (s To n)$   
**!**  $\vdash 0! = 1 \wedge (\forall m \bullet (m + 1)! = (m + 1) * m!)$

**Exp**  $\vdash ConstSpec$   
 $(\lambda Exp'$   
 $\bullet Exp' 0. = 1. \wedge (\forall x \bullet (Exp' Deriv Exp' x) x))$   
 $Exp$

**Log**  $\vdash ConstSpec (\lambda Log' \bullet \forall x \bullet Log' (Exp x) = x) Log$

**Sin**  
**Cos**  $\vdash ConstSpec$   
 $(\lambda (Sin', Cos')$   
 $\bullet Sin' 0. = 0.$   
 $\wedge Cos' 0. = 1.$   
 $\wedge (\forall x \bullet (Sin' Deriv Cos' x) x)$   
 $\wedge (\forall x \bullet (Cos' Deriv \sim (Sin' x)) x))$   
 $(Sin, Cos)$

**ArchimedesConstant**  
 $\vdash ConstSpec$   
 $(\lambda ArchimedesConstant'$   
 $\bullet 0. < ArchimedesConstant'$   
 $\wedge (\forall x$   
 $\bullet Sin x = 0.$   
 $\Leftrightarrow (\exists m \bullet x = \mathbb{N}R m * ArchimedesConstant')$   
 $\vee (\exists m$   
 $\bullet x = \sim (\mathbb{N}R m * ArchimedesConstant'))))$

$\pi$   
 $\dagger \# \dashrightarrow$   $\vdash \forall f c a$   
 $\bullet (f \dagger \# \dashrightarrow c) a$   
 $\Leftrightarrow (\forall e$   
 $\bullet 0. < e$   
 $\Rightarrow (\exists b$   
 $\bullet a < b$   
 $\wedge (\forall x$   
 $\bullet a < x \wedge x < b \Rightarrow Abs (f x - c) < e)))$

$\dashrightarrow \dagger \# >$   $\vdash \forall f c$

$\bullet f \text{ --+\#> } c$   
 $\Leftrightarrow (\forall e$   
 $\bullet 0. < e \Rightarrow (\exists b \bullet \forall x \bullet b < x \Rightarrow \text{Abs } (f \ x - c) < e))$   
**--+\#>+\#**  $\vdash \forall f \bullet f \text{ --+\#>+\#} \Leftrightarrow (\forall x \bullet \exists b \bullet \forall y \bullet b < y \Rightarrow x < f \ y)$   
**Sqrt**  $\vdash \text{ConstSpec}$   
 $(\lambda \text{ Sqrt}'$   
 $\bullet \forall x$   
 $\bullet 0. \leq x$   
 $\Rightarrow 0. \leq \text{Sqrt}' \ x$   
 $\wedge \text{Sqrt}' \ x \wedge 2 = x$   
 $\wedge \text{Sqrt}' \ \text{Cts } x)$   
*Sqrt*  
**Tan**  $\vdash \forall x \bullet \text{Tan } x = \text{Sin } x * \text{Cos } x^{-1}$   
**Cotan**  $\vdash \forall x \bullet \text{Cotan } x = \text{Cos } x * \text{Sin } x^{-1}$   
**Sec**  $\vdash \forall x \bullet \text{Sec } x = \text{Cos } x^{-1}$   
**Cosec**  $\vdash \forall x \bullet \text{Cosec } x = \text{Sin } x^{-1}$   
**ArcSin**  $\vdash \text{ConstSpec}$   
 $(\lambda \text{ ArcSin}'$   
 $\bullet (\forall x \bullet \text{Abs } x \leq 1 / 2 * \pi \Rightarrow \text{ArcSin}' (\text{Sin } x) = x)$   
 $\wedge (\forall x$   
 $\bullet \text{Abs } x \leq 1.$   
 $\Rightarrow \text{Sin } (\text{ArcSin}' \ x) = x \wedge \text{ArcSin}' \ \text{Cts } x))$   
*ArcSin*  
**ArcCos**  $\vdash \text{ConstSpec}$   
 $(\lambda \text{ ArcCos}'$   
 $\bullet (\forall x \bullet 0. \leq x \wedge x \leq \pi \Rightarrow \text{ArcCos}' (\text{Cos } x) = x)$   
 $\wedge (\forall x$   
 $\bullet \text{Abs } x \leq 1.$   
 $\Rightarrow \text{Cos } (\text{ArcCos}' \ x) = x \wedge \text{ArcCos}' \ \text{Cts } x))$   
*ArcCos*  
**Sinh**  $\vdash \forall x \bullet \text{Sinh } x = 1 / 2 * (\text{Exp } x - \text{Exp } (\sim x))$   
**Cosh**  $\vdash \forall x \bullet \text{Cosh } x = 1 / 2 * (\text{Exp } x + \text{Exp } (\sim x))$   
**Tanh**  $\vdash \forall x \bullet \text{Tanh } x = \text{Sinh } x * \text{Cosh } x^{-1}$   
**Cotanh**  $\vdash \forall x \bullet \text{Cotanh } x = \text{Cosh } x * \text{Sinh } x^{-1}$   
**Sech**  $\vdash \forall x \bullet \text{Sech } x = \text{Cosh } x^{-1}$   
**Cosech**  $\vdash \forall x \bullet \text{Cosech } x = \text{Sinh } x^{-1}$   
**ArcSinh**  $\vdash \text{ConstSpec}$   
 $(\lambda \text{ ArcSinh}'$   
 $\bullet \forall x$   
 $\bullet \text{ArcSinh}' (\text{Sinh } x) = x$   
 $\wedge \text{Sinh } (\text{ArcSinh}' \ x) = x$   
 $\wedge \text{ArcSinh}' \ \text{Cts } x)$   
*ArcSinh*  
**TaggedPartition**  $\vdash \forall t \ I \ n$   
 $\bullet (t, I, n) \in \text{TaggedPartition}$   
 $\Leftrightarrow (\forall m \bullet m < n \Rightarrow I \ m < I \ (m + 1))$   
 $\wedge (\forall m$   
 $\bullet m < n$   
 $\Rightarrow t \ m \in \text{ClosedInterval } (I \ m) (I \ (m + 1)))$   
**RiemannSum**  $\vdash \forall f \ t \ I \ n$

**Gauge**  
**Fine**

- $RiemannSum\ f\ (t, I, n)$   
 $= Sigma\ (\lambda\ m\bullet\ f\ (t\ m) * (I\ (m + 1) - I\ m))\ n$
- $\vdash \forall G\bullet\ G \in Gauge \Leftrightarrow (\forall x\bullet\ G\ x \in Open_R \wedge x \in G\ x)$
- $\vdash \forall t\ I\ n\ G$ 
  - $(t, I, n) \in G\ Fine$   
 $\Leftrightarrow (\forall m$ 
    - $m < n$   
 $\Rightarrow ClosedInterval\ (I\ m)\ (I\ (m + 1)) \subseteq G\ (t\ m))$

**Int<sub>R</sub>**

- $\vdash \forall f\ c$ 
  - $f\ Int_R\ c$   
 $\Leftrightarrow (\forall e$ 
    - $0. < e$   
 $\Rightarrow (\exists G\ a\ b$ 
      - $G \in Gauge$   
 $\wedge a < b$   
 $\wedge (\forall t\ I\ n$ 
        - $(t, I, n) \in TaggedPartition$   
 $\wedge I\ 0 < a$   
 $\wedge b < I\ n$   
 $\wedge (t, I, n) \in G\ Fine$   
 $\Rightarrow Abs\ (RiemannSum\ f\ (t, I, n) - c)$   
 $< e)))$

**CharFun**  
**Int**  
**Area**

- $\vdash \forall A\ x\bullet\ \chi\ A\ x = (if\ x \in A\ then\ 1.\ else\ 0.)$
- $\vdash \forall f\ c\ A\bullet\ (f\ Int\ c)\ A \Leftrightarrow (\lambda\ x\bullet\ \chi\ A\ x * f\ x)\ Int_R\ c$
- $\vdash \forall A\ c$ 
  - $A\ Area\ c$   
 $\Leftrightarrow (\exists\ sf$ 
    - $sf\ Int_R\ c$   
 $\wedge (\forall x\bullet\ \chi\ \{y | (x, y) \in A\}\ Int_R\ sf\ x))$

## A.6 Theorems

### *PlusCoeffs.consistent*

$\vdash Consistent$   
 $(\lambda\ PlusCoeffs'$ 

- $(\forall l\bullet\ PlusCoeffs'\ \ []\ l = l)$   
 $\wedge (\forall l\bullet\ PlusCoeffs'\ l\ \ [] = l)$   
 $\wedge (\forall c1\ l1\ c2\ l2$ 
  - $PlusCoeffs'\ (Cons\ c1\ l1)\ (Cons\ c2\ l2)$   
 $= Cons\ (c1 + c2)\ (PlusCoeffs'\ l1\ l2)))$

### *plus\_coeffs\_def*

$\vdash (\forall l\bullet\ PlusCoeffs\ \ []\ l = l)$   
 $\wedge (\forall l\bullet\ PlusCoeffs\ l\ \ [] = l)$   
 $\wedge (\forall c1\ l1\ c2\ l2$ 

- $PlusCoeffs\ (Cons\ c1\ l1)\ (Cons\ c2\ l2)$   
 $= Cons\ (c1 + c2)\ (PlusCoeffs\ l1\ l2))$

### *const\_eval\_thm*

$\vdash \forall c\bullet\ (\lambda\ x\bullet\ c) = PolyEval\ [c]$

### *id\_eval\_thm*

$\vdash (\lambda\ x\bullet\ x) = PolyEval\ [0.; 1.]$

### *plus\_eval\_thm*

$\vdash \forall l1\ l2$

$\bullet (\lambda x \bullet \text{PolyEval } l1 \ x + \text{PolyEval } l2 \ x)$   
 $= \text{PolyEval } (\text{PlusCoeffs } l1 \ l2)$

**const\_times\_eval\_thm**

$\vdash \forall c \ l$

$\bullet (\lambda x \bullet c * \text{PolyEval } l \ x)$   
 $= \text{PolyEval } (\text{Map } (\lambda y \bullet c * y) \ l)$

**times\_eval\_thm**

$\vdash \forall l1 \ l2$

$\bullet (\lambda x \bullet \text{PolyEval } l1 \ x * \text{PolyEval } l2 \ x)$   
 $= \text{PolyEval } (\text{TimesCoeffs } l1 \ l2)$

**poly\_induction\_thm**

$\vdash \forall p$

$\bullet (\forall c \bullet p (\lambda x \bullet c))$   
 $\wedge p (\lambda x \bullet x)$   
 $\wedge (\forall f \ g \bullet p \ f \wedge p \ g \Rightarrow p (\lambda x \bullet f \ x + g \ x))$   
 $\wedge (\forall f \ g \bullet p \ f \wedge p \ g \Rightarrow p (\lambda x \bullet f \ x * g \ x))$   
 $\Rightarrow (\forall h \bullet h \in \text{PolyFunc} \Rightarrow p \ h)$

**poly\_func\_eq\_poly\_eval\_thm**

$\vdash \text{PolyFunc} = \{f \mid \exists l \bullet f = \text{PolyEval } l\}$

**const\_poly\_func\_thm**

$\vdash \forall c \bullet (\lambda x \bullet c) \in \text{PolyFunc}$

**id\_poly\_func\_thm**

$\vdash (\lambda x \bullet x) \in \text{PolyFunc}$

**plus\_poly\_func\_thm**

$\vdash \forall f \ g$

$\bullet f \in \text{PolyFunc} \wedge g \in \text{PolyFunc}$   
 $\Rightarrow (\lambda x \bullet f \ x + g \ x) \in \text{PolyFunc}$

**times\_poly\_func\_thm**

$\vdash \forall f \ g$

$\bullet f \in \text{PolyFunc} \wedge g \in \text{PolyFunc}$   
 $\Rightarrow (\lambda x \bullet f \ x * g \ x) \in \text{PolyFunc}$

**comp\_poly\_func\_thm**

$\vdash \forall f \ g$

$\bullet f \in \text{PolyFunc} \wedge g \in \text{PolyFunc}$   
 $\Rightarrow (\lambda x \bullet f (g \ x)) \in \text{PolyFunc}$

**poly\_eval\_append\_thm**

$\vdash \forall l1 \ l2 \ x$

$\bullet \text{PolyEval } (l1 \ \wedge \ l2) \ x$   
 $= \text{PolyEval } l1 \ x + x \ \wedge \ \# \ l1 * \text{PolyEval } l2 \ x$

**poly\_eval\_rev\_thm**

$\vdash (\forall x \bullet \text{PolyEval } (\text{Rev } []) \ x = 0.)$

$\wedge (\forall c \ l \ x$

$\bullet \text{PolyEval } (\text{Rev } (\text{Cons } c \ l)) \ x$   
 $= c * x \ \wedge \ \# \ l + \text{PolyEval } (\text{Rev } l) \ x)$

**poly\_diff\_powers\_thm**

$\vdash \forall n \ x \ y$

$\bullet (x - y)$   
 $* \text{PolyEval } (\text{Rev } ((\lambda m \bullet y \ \wedge \ m) \ \text{To } (n + 1))) \ x$   
 $= x \ \wedge \ (n + 1) - y \ \wedge \ (n + 1)$

**length\_plus\_coeffs\_thm**

$\vdash \forall l1 \ l2$

- $\#$  (*PlusCoeffs*  $l1$   $l2$ )  
= (if  $\# l2 < \# l1$  then  $\# l1$  else  $\# l2$ )

***poly\_linear\_div\_thm***

- $\vdash \forall l1\ c$
- $\neg l1 = []$   
 $\Rightarrow (\exists l2\ r$
- $\# l2 < \# l1$   
 $\wedge (\lambda x \bullet \text{PolyEval } l1\ x)$   
 $= (\lambda x \bullet (x - c) * \text{PolyEval } l2\ x + r))$

***poly\_remainder\_thm***

- $\vdash \forall l1\ c$
- $\neg l1 = []$   
 $\Rightarrow (\exists l2$
- $\# l2 < \# l1$   
 $\wedge (\lambda x \bullet \text{PolyEval } l1\ x)$   
 $= (\lambda x$
- $(x - c) * \text{PolyEval } l2\ x + \text{PolyEval } l1\ c))$

***poly\_factor\_thm***

- $\vdash \forall l1\ c$
- $\neg l1 = [] \wedge \text{PolyEval } l1\ c = 0.$   
 $\Rightarrow (\exists l2$
- $\# l2 < \# l1$   
 $\wedge (\lambda x \bullet \text{PolyEval } l1\ x)$   
 $= (\lambda x \bullet (x - c) * \text{PolyEval } l2\ x))$

***poly\_roots\_finite\_thm***

- $\vdash \forall f\ c \bullet f \in \text{PolyFunc} \wedge \neg f\ c = 0. \Rightarrow \text{Roots } f \in \text{Finite}$

**$\mathbb{R}$ \_0\_≤\_abs\_thm**

- $\vdash \forall x \bullet 0. \leq \text{Abs } x$

**$\mathbb{R}$ \_abs\_plus\_thm**

- $\vdash \forall x\ y \bullet \text{Abs } (x + y) \leq \text{Abs } x + \text{Abs } y$

**$\mathbb{R}$ \_abs\_subtract\_thm**

- $\vdash \forall x\ y \bullet \text{Abs } (x - y) \leq \text{Abs } x + \text{Abs } y$

**$\mathbb{R}$ \_abs\_plus\_minus\_thm**

- $\vdash \forall x\ y \bullet \text{Abs } (x + \sim y) \leq \text{Abs } x + \text{Abs } y$

**$\mathbb{R}$ \_abs\_diff\_bounded\_thm**

- $\vdash \forall x\ y\ z$
- $0. < z$   
 $\Rightarrow (\text{Abs } (x + \sim y) < z \Leftrightarrow y + \sim z < x \wedge x < y + z)$

**$\mathbb{R}$ \_abs\_plus\_bc\_thm**

- $\vdash \forall x\ y\ z \bullet \text{Abs } x \leq \text{Abs } (y + z) \Rightarrow \text{Abs } x \leq \text{Abs } y + \text{Abs } z$

**$\mathbb{R}$ \_abs\_abs\_minus\_thm**

- $\vdash \forall x\ y \bullet \text{Abs } (\text{Abs } x - \text{Abs } y) \leq \text{Abs } (x - y)$

**$\mathbb{R}$ \_abs\_abs\_thm**

- $\vdash \forall x \bullet \text{Abs } (\text{Abs } x) = \text{Abs } x$

**$\mathbb{R}$ \_abs\_times\_thm**

- $\vdash \forall x\ y \bullet \text{Abs } (x * y) = \text{Abs } x * \text{Abs } y$

**$\mathbb{R}$ \_abs\_ℝ\_N\_exp\_thm**

- $\vdash \forall x\ m \bullet \text{Abs } (x \wedge m) = \text{Abs } x \wedge m$

**$\mathbb{R}$ \_abs\_eq\_0\_thm**

- $\vdash \forall x \bullet \text{Abs } x = 0. \Leftrightarrow x = 0.$

**$\mathbb{R}$ \_abs\_eq\_0\_thm1**

$\vdash \forall x \bullet x = 0. \Leftrightarrow \text{Abs } x = 0.$   
 **$\mathbb{R}_{\text{abs\_}\leq\text{0\_thm}}$**   
 $\vdash \forall x \bullet \text{Abs } x \leq 0. \Leftrightarrow x = 0.$   
 **$\mathbb{R}_{\text{abs\_0\_thm}}$**   $\vdash \text{Abs } 0. = 0.$   
 **$\mathbb{R}_{\text{abs\_recip\_thm}}$**   
 $\vdash \forall x \bullet \neg x = 0. \Rightarrow \text{Abs } (x^{-1}) = \text{Abs } x^{-1}$   
 **$\mathbb{R}_{\text{abs\_squared\_thm}}$**   
 $\vdash \forall x \bullet \text{Abs } x^{\wedge} 2 = x^{\wedge} 2$   
 **$\mathbb{R}_{\text{abs\_less\_times\_thm}}$**   
 $\vdash \forall x \ t \ y \ u$   
 $\bullet \text{Abs } x < t \wedge \text{Abs } y < u \Rightarrow \text{Abs } x * \text{Abs } y < t * u$   
 **$\mathbb{R}_{\neg\text{0\_abs\_thm}}$**   
 $\vdash \forall x \bullet \neg x = 0. \Leftrightarrow 0. < \text{Abs } x$   
 **$\mathbb{R}_{\text{less\_recip\_less\_thm}}$**   
 $\vdash \forall x \ y \bullet 0. < x \wedge x < y \Rightarrow y^{-1} < x^{-1}$   
 **$\mathbb{R}_{\text{abs\_}\leq\text{less\_interval\_thm}}$**   
 $\vdash \forall x \ y$   
 $\bullet (\text{Abs } x \leq y \Leftrightarrow x \in \text{ClosedInterval } (\sim y) \ y)$   
 $\wedge (\text{Abs } x < y \Leftrightarrow x \in \text{OpenInterval } (\sim y) \ y)$   
 **$\mathbb{R}_{\text{plus\_recip\_thm}}$**   
 $\vdash \forall x \ y$   
 $\bullet \neg x = 0. \wedge \neg y = 0.$   
 $\Rightarrow x^{-1} + y^{-1} = (x + y) * x^{-1} * y^{-1}$   
 **$\text{NR}_{\text{recip\_thm}}$**   $\vdash \forall m \bullet \text{NR } (m + 1)^{-1}^{-1} = \text{NR } (m + 1)$   
 **$\text{NR}_{\text{0\_less\_recip\_thm}}$**   
 $\vdash \forall m \bullet 0. < \text{NR } (m + 1)^{-1}$   
 **$\text{NR}_{\text{recip\_not\_eq\_0\_thm}}$**   
 $\vdash \forall m \bullet \neg \text{NR } (m + 1)^{-1} = 0.$   
 **$\mathbb{R}_{\text{N\_exp\_0\_1\_thm}}$**   
 $\vdash \forall m \bullet 0.^{\wedge} (m + 1) = 0. \wedge 1.^{\wedge} m = 1.$   
 **$\mathbb{R}_{\text{N\_exp\_square\_thm}}$**   
 $\vdash \forall x \bullet x^{\wedge} 2 = x * x$   
 **$\mathbb{R}_{\text{N\_exp\_0\_less\_thm}}$**   
 $\vdash \forall m \ x \bullet 0. < x \Rightarrow 0. < x^{\wedge} m$   
 **$\mathbb{R}_{\text{N\_exp\_1\_less\_mono\_thm}}$**   
 $\vdash \forall x \ m \bullet 1. < x \Rightarrow x^{\wedge} m < x^{\wedge} (m + 1)$   
 **$\mathbb{R}_{\text{N\_exp\_1\_less\_mono\_thm1}}$**   
 $\vdash \forall x \ m \ n \bullet 1. < x \wedge m < n \Rightarrow x^{\wedge} m < x^{\wedge} n$   
 **$\mathbb{R}_{\leq\text{times\_mono\_thm}}$**   
 $\vdash \forall x \ y \ z \bullet 0. \leq x \wedge y \leq z \Rightarrow x * y \leq x * z$   
 **$\mathbb{R}_{\text{N\_exp\_}\neg\text{eq\_0\_thm}}$**   
 $\vdash \forall m \ x \bullet \neg x = 0. \Rightarrow \neg x^{\wedge} m = 0.$   
 **$\mathbb{R}_{\text{N\_exp\_plus\_thm}}$**   
 $\vdash \forall x \ m \ n \bullet x^{\wedge} (m + n) = x^{\wedge} m * x^{\wedge} n$   
 **$\mathbb{R}_{\text{N\_exp\_times\_thm}}$**   
 $\vdash \forall x \ y \ m \bullet (x * y)^{\wedge} m = x^{\wedge} m * y^{\wedge} m$   
 **$\mathbb{R}_{\text{N\_exp\_recip\_thm}}$**   
 $\vdash \forall m \ x \bullet \neg x = 0. \Rightarrow (x^{\wedge} m)^{-1} = (x^{-1})^{\wedge} m$   
 **$\mathbb{R}_{\text{N\_exp\_recip\_thm1}}$**   
 $\vdash \forall m \ x \bullet \neg x = 0. \Rightarrow (x^{-1})^{\wedge} m = (x^{\wedge} m)^{-1}$   
 **$\mathbb{R}_{\text{N\_exp\_1\_}\leq\text{thm}}$**

$\vdash \forall m x \bullet 1. \leq x \Rightarrow 1. \leq x \wedge m$   
 **$\mathbb{R}_N$ .exp\_less\_1\_mono\_thm**  
 $\vdash \forall x m \bullet 0. < x \wedge x < 1. \Rightarrow x \wedge (m + 1) < x \wedge m$   
 **$\mathbb{R}_N$ .exp\_less\_mono\_thm**  
 $\vdash \forall x m \bullet 0. < x \Rightarrow 0. < x \wedge m$   
 **$\mathbb{R}_N$ .exp\_less\_1\_mono\_thm1**  
 $\vdash \forall x m n \bullet 0. < x \wedge x < 1. \wedge m < n \Rightarrow x \wedge n < x \wedge m$   
 **$\mathbb{R}_N$ .exp\_linear\_estimate\_thm**  
 $\vdash \forall x m \bullet 0. < x \Rightarrow 1. + \mathbb{N}R m * x \leq (1. + x) \wedge m$   
 **$\mathbb{R}_0$ .<\_square\_thm**  
 $\vdash \forall x \bullet 0. \leq x \wedge 2$   
 **$\mathbb{R}$ .square\_eq\_0\_thm**  
 $\vdash \forall x \bullet x \wedge 2 = 0. \Leftrightarrow x = 0.$   
 **$\mathbb{R}$ .bound\_below\_2\_thm**  
 $\vdash \forall x y$   
 $\bullet 0. < x \wedge 0. < y \Rightarrow (\exists d \bullet 0. < d \wedge d < x \wedge d < y)$   
 **$\mathbb{R}$ .bound\_below\_3\_thm**  
 $\vdash \forall x y z$   
 $\bullet 0. < x \wedge 0. < y \wedge 0. < z$   
 $\Rightarrow (\exists d \bullet 0. < d \wedge d < x \wedge d < y \wedge d < z)$   
 **$\mathbb{R}$ .max\_2\_thm**  $\vdash \forall x y \bullet \exists z \bullet x < z \wedge y < z$   
 **$\mathbb{R}$ .max\_3\_thm**  $\vdash \forall x y z \bullet \exists t \bullet x < t \wedge y < t \wedge z < t$   
 **$\mathbb{R}$ .min\_2\_thm**  $\vdash \forall x y \bullet \exists z \bullet z < x \wedge z < y$   
 **$\mathbb{R}$ .min\_3\_thm**  $\vdash \forall x y z \bullet \exists t \bullet t < x \wedge t < y \wedge t < z$   
 **$\mathbb{R}$ .archimedean\_thm**  
 $\vdash \forall x \bullet \exists m \bullet x < \mathbb{N}R m$   
 **$\mathbb{R}$ .archimedean\_recip\_thm**  
 $\vdash \forall x \bullet 0. < x \Rightarrow (\exists m \bullet \mathbb{N}R (m + 1)^{-1} < x)$   
 **$\mathbb{R}$ .archimedean\_times\_thm**  
 $\vdash \forall x y \bullet 0. < x \Rightarrow (\exists m \bullet y < \mathbb{N}R m * x)$   
 **$\mathbb{R}$ .archimedean\_division\_thm**  
 $\vdash \forall d y$   
 $\bullet 0. < d \wedge 0. \leq y$   
 $\Rightarrow (\exists q r \bullet y = \mathbb{N}R q * d + r \wedge 0. \leq r \wedge r < d)$   
 **$\mathbb{R}_N$ .exp\_tends\_to\_infinity\_thm**  
 $\vdash \forall x y \bullet 1. < x \Rightarrow (\exists m \bullet y < x \wedge m)$   
 **$\mathbb{R}_N$ .exp\_tends\_to\_0\_thm**  
 $\vdash \forall x y \bullet 0. < x \wedge x < 1. \wedge 0. < y \Rightarrow (\exists m \bullet x \wedge m < y)$   
**const\_lim\_seq\_thm**  
 $\vdash \forall c \bullet (\lambda m \bullet c) \rightarrow c$   
**plus\_lim\_seq\_thm**  
 $\vdash \forall s1 c1 s2 c2$   
 $\bullet s1 \rightarrow c1 \wedge s2 \rightarrow c2 \Rightarrow (\lambda m \bullet s1 m + s2 m) \rightarrow c1 + c2$   
**lim\_seq\_bounded\_thm**  
 $\vdash \forall s c \bullet s \rightarrow c \Rightarrow (\exists b \bullet 0. < b \wedge (\forall m \bullet \text{Abs } (s m) < b))$   
**times\_lim\_seq\_thm**  
 $\vdash \forall s1 c1 s2 c2$   
 $\bullet s1 \rightarrow c1 \wedge s2 \rightarrow c2 \Rightarrow (\lambda m \bullet s1 m * s2 m) \rightarrow c1 * c2$   
**minus\_lim\_seq\_thm**  
 $\vdash \forall s c \bullet s \rightarrow c \Leftrightarrow (\lambda m \bullet \sim (s m)) \rightarrow \sim c$   
**poly\_lim\_seq\_thm**



$\vdash \forall f s t$   
 $\bullet f \in \text{PolyFunc} \wedge s \rightarrow t \Rightarrow (\lambda x \bullet f (s x)) \rightarrow f t$   
*recip\_lim\_seq\_thm*  
 $\vdash \forall s t \bullet s \rightarrow t \wedge \neg t = 0. \Rightarrow (\lambda m \bullet s m^{-1}) \rightarrow t^{-1}$   
*lim\_seq\_choice\_thm*  
 $\vdash \forall p s1 s2 x$   
 $\bullet s1 \rightarrow x \wedge s2 \rightarrow x$   
 $\Rightarrow (\lambda m \bullet \text{if } p m \text{ then } s1 m \text{ else } s2 m) \rightarrow x$   
*abs\_lim\_seq\_thm*  
 $\vdash \forall s x \bullet s \rightarrow x \Rightarrow (\lambda m \bullet \text{Abs } (s m)) \rightarrow \text{Abs } x$   
*lim\_seq\_recip\_N\_thm*  
 $\vdash \forall x \bullet (\lambda m \bullet x + \mathbb{N} (m + 1)^{-1}) \rightarrow x$   
*lim\_seq\_not\_eq\_thm*  
 $\vdash \forall x \bullet \exists s \bullet s \rightarrow x \wedge (\forall m \bullet \neg s m = x)$   
*lim\_seq\_shift\_thm*  
 $\vdash \forall m s x \bullet s \rightarrow x \Leftrightarrow (\lambda n \bullet s (n + m)) \rightarrow x$   
*lim\_seq\_not\_eq\_thm1*  
 $\vdash \forall x d$   
 $\bullet 0. < d$   
 $\Rightarrow (\exists s$   
 $\bullet s \rightarrow x \wedge (\forall m \bullet \text{Abs } (s m - x) < d \wedge \neg s m = x))$   
*lim\_seq\_not\_0\_thm*  
 $\vdash \forall s x$   
 $\bullet s \rightarrow x \wedge \neg x = 0. \Rightarrow (\exists n \bullet \forall m \bullet n \leq m \Rightarrow \neg s m = 0.)$   
*lim\_seq\_upper\_bound\_thm*  
 $\vdash \forall s b c \bullet (\forall m \bullet s m \leq b) \wedge s \rightarrow c \Rightarrow c \leq b$   
*lim\_seq\_unique\_thm*  
 $\vdash \forall s b c \bullet s \rightarrow b \wedge s \rightarrow c \Rightarrow b = c$   
*lim\_seq\_diffs\_tend\_to\_0\_thm*  
 $\vdash \forall s c \bullet s \rightarrow c \Rightarrow (\lambda m \bullet s (m + 1) - s m) \rightarrow 0.$   
*cts\_lim\_seq\_thm*  
 $\vdash \forall f x$   
 $\bullet f \text{ Cts } x \Leftrightarrow (\forall s \bullet s \rightarrow x \Rightarrow (\lambda m \bullet f (s m)) \rightarrow f x)$   
*cts\_lim\_seq\_thm1*  
 $\vdash \forall f x$   
 $\bullet f \text{ Cts } x$   
 $\Leftrightarrow (\forall s$   
 $\bullet s \rightarrow x \wedge (\forall m \bullet \neg s m = x)$   
 $\Rightarrow (\lambda m \bullet f (s m)) \rightarrow f x)$   
*cts\_lim\_seq\_thm2*  
 $\vdash \forall f x$   
 $\bullet f \text{ Cts } x$   
 $\Leftrightarrow (\exists a b$   
 $\bullet a < x$   
 $\wedge x < b$   
 $\wedge (\forall s$   
 $\bullet s \rightarrow x$   
 $\wedge (\forall m \bullet \neg s m = x \wedge a < s m \wedge s m < b)$   
 $\Rightarrow (\lambda m \bullet f (s m)) \rightarrow f x)$   
*cts\_local\_thm*  
 $\vdash \forall f g x a b$

- $a < x$
- $\wedge x < b$
- $\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow f y = g y)$
- $\wedge g \text{ Cts } x$
- $\Rightarrow f \text{ Cts } x$

**const\_cts\_thm**

$$\vdash \forall c t \bullet (\lambda x \bullet c) \text{ Cts } t$$

**id\_cts\_thm**

$$\vdash \forall t \bullet (\lambda x \bullet x) \text{ Cts } t$$

**plus\_cts\_thm**

$$\vdash \forall f g t \bullet f \text{ Cts } t \wedge g \text{ Cts } t \Rightarrow (\lambda x \bullet f x + g x) \text{ Cts } t$$

**times\_cts\_thm**

$$\vdash \forall f g t \bullet f \text{ Cts } t \wedge g \text{ Cts } t \Rightarrow (\lambda x \bullet f x * g x) \text{ Cts } t$$

**poly\_cts\_thm**

$$\vdash \forall f t \bullet f \in \text{PolyFunc} \Rightarrow f \text{ Cts } t$$

**comp\_cts\_thm**

$$\vdash \forall f g t \bullet f \text{ Cts } g t \wedge g \text{ Cts } t \Rightarrow (\lambda x \bullet f (g x)) \text{ Cts } t$$

**minus\_cts\_thm**

$$\vdash \forall t \bullet \sim \text{ Cts } t$$

**minus\_comp\_cts\_thm**

$$\vdash \forall f t \bullet f \text{ Cts } t \Rightarrow (\lambda x \bullet \sim (f x)) \text{ Cts } t$$

**recip\_cts\_thm**

$$\vdash \forall t \bullet \neg t = 0. \Rightarrow \$^{-1} \text{ Cts } t$$

**recip\_comp\_cts\_thm**

$$\vdash \forall f t \bullet f \text{ Cts } t \wedge \neg f t = 0. \Rightarrow (\lambda x \bullet f x^{-1}) \text{ Cts } t$$

**$\mathbb{R}_N$ \_exp\_cts\_thm**

$$\vdash \forall m t \bullet (\lambda x \bullet x \wedge m) \text{ Cts } t$$

**$\mathbb{R}_N$ \_exp\_comp\_cts\_thm**

$$\vdash \forall f m t \bullet f \text{ Cts } t \Rightarrow (\lambda x \bullet f x \wedge m) \text{ Cts } t$$

**abs\_cts\_thm**

$$\vdash \forall t \bullet \text{Abs} \text{ Cts } t$$

**abs\_comp\_cts\_thm**

$$\vdash \forall f m t \bullet f \text{ Cts } t \Rightarrow (\lambda x \bullet \text{Abs} (f x)) \text{ Cts } t$$

**cts\_extension\_thm1**

$$\begin{aligned} &\vdash \forall a b f \\ &\bullet a < b \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x) \\ &\Rightarrow (\exists g \\ &\bullet (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow g x = f x) \\ &\quad \wedge (\forall x \bullet x < a \Rightarrow g x = f a) \\ &\quad \wedge (\forall x \bullet b < x \Rightarrow g x = f b) \\ &\quad \wedge (\forall x \bullet g \text{ Cts } x)) \end{aligned}$$

**cts\_extension\_thm**

$$\begin{aligned} &\vdash \forall a b f \\ &\bullet a < b \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x) \\ &\Rightarrow (\exists g \\ &\bullet (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow g x = f x) \\ &\quad \wedge (\forall x \bullet g \text{ Cts } x)) \end{aligned}$$

**open\_ $\mathbb{R}$ \_delta\_thm**

$$\begin{aligned} &\vdash \forall A \\ &\bullet A \in \text{Open}_R \\ &\Leftrightarrow (\forall t \\ &\bullet t \in A \\ &\Rightarrow (\exists d \\ &\bullet 0. < d \wedge (\forall y \bullet \text{Abs} (y - t) < d \Rightarrow y \in A))) \end{aligned}$$

**lim\_seq\_open\_ $\mathbb{R}$ \_thm**

$$\vdash \forall s x$$

- $s \rightarrow x$
- $\Leftrightarrow (\forall A$
- $A \in \text{Open}_R \wedge x \in A$
- $\Rightarrow (\exists n \bullet \forall m \bullet n \leq m \Rightarrow s m \in A))$

*cts\_open\_R\_thm*

- $\vdash \forall f$
- $(\forall x \bullet f \text{ Cts } x)$
- $\Leftrightarrow (\forall A \bullet A \in \text{Open}_R \Rightarrow \{x \mid f x \in A\} \in \text{Open}_R)$

*closed\_interval\_closed\_thm*

- $\vdash \forall x y \bullet \text{ClosedInterval } x y \in \text{Closed}_R$

*cts\_estimate\_thm*

- $\vdash \forall f x \text{ lb } \text{ub}$
- $f \text{ Cts } x$
- $\wedge (\forall d$
- $0. < d$
- $\Rightarrow (\exists z \bullet \text{Abs } (z - x) < d \wedge \text{lb} \leq f z)$
- $\wedge (\exists z \bullet \text{Abs } (z - x) < d \wedge f z \leq \text{ub}))$
- $\Rightarrow \text{lb} \leq f x \wedge f x \leq \text{ub}$

*cts\_estimate\_0\_thm*

- $\vdash \forall f x$
- $f \text{ Cts } x$
- $\wedge (\forall d$
- $0. < d$
- $\Rightarrow (\exists z \bullet \text{Abs } (z - x) < d \wedge 0. \leq f z)$
- $\wedge (\exists z \bullet \text{Abs } (z - x) < d \wedge f z \leq 0.))$
- $\Rightarrow f x = 0.$

*cts\_limit\_unique\_thm*

- $\vdash \forall f g x a b$
- $a < x$
- $\wedge x < b$
- $\wedge f \text{ Cts } x$
- $\wedge g \text{ Cts } x$
- $\wedge (\forall y \bullet a < y \wedge y < b \wedge \neg y = x \Rightarrow f y = g y)$
- $\Rightarrow f x = g x$

*cts\_open\_interval\_thm*

- $\vdash \forall f a b x$
- $f x \in \text{OpenInterval } a b$
- $\wedge (\forall c d$
- $f x \in \text{OpenInterval } c d$
- $\wedge \text{ClosedInterval } c d \subseteq \text{OpenInterval } a b$
- $\Rightarrow (\exists s t$
- $x \in \text{OpenInterval } s t$
- $\wedge (\forall y$
- $y \in \text{OpenInterval } s t$
- $\Rightarrow f y \in \text{OpenInterval } c d))$
- $\Rightarrow f \text{ Cts } x$

*darboux\_cts\_mono\_thm*

- $\vdash \forall f a b x$
- $(\forall x y$
- $x \in \text{ClosedInterval } a b$
- $\wedge y \in \text{ClosedInterval } a b$

$$\begin{aligned}
& \wedge x < y \\
& \Rightarrow f x < f y) \\
& \wedge (\forall y \\
& \bullet y \in \text{OpenInterval } (f a) (f b) \\
& \Rightarrow (\exists x \bullet a < x \wedge x < b \wedge f x = y)) \\
& \wedge x \in \text{OpenInterval } a b \\
& \Rightarrow f \text{ Cts } x
\end{aligned}$$

**darboux\_cts\_mono\_thm1**

$$\begin{aligned}
& \vdash \forall f a b x \\
& \bullet (\forall x y \\
& \bullet x \in \text{OpenInterval } a b \\
& \quad \wedge y \in \text{OpenInterval } a b \\
& \quad \wedge x < y \\
& \quad \Rightarrow f x < f y) \\
& \wedge (\forall x y z \\
& \bullet x \in \text{OpenInterval } a b \\
& \quad \wedge y \in \text{OpenInterval } a b \\
& \quad \wedge z \in \text{OpenInterval } (f x) (f y) \\
& \quad \Rightarrow (\exists t \bullet t \in \text{OpenInterval } a b \wedge f t = z)) \\
& \wedge x \in \text{OpenInterval } a b \\
& \Rightarrow f \text{ Cts } x
\end{aligned}$$

**caratheodory\_deriv\_thm**

$$\begin{aligned}
& \vdash \forall f c x \\
& \bullet (f \text{ Deriv } c) x \\
& \Leftrightarrow (\exists g \\
& \bullet (\forall y \bullet f y - f x = g y * (y - x)) \\
& \quad \wedge g \text{ Cts } x \\
& \quad \wedge g x = c)
\end{aligned}$$

**deriv\_cts\_thm**

$$\vdash \forall f c x \bullet (f \text{ Deriv } c) x \Rightarrow f \text{ Cts } x$$

**const\_deriv\_thm**

$$\vdash \forall c t \bullet ((\lambda x \bullet c) \text{ Deriv } 0.) t$$

**id\_deriv\_thm**  $\vdash \forall t \bullet ((\lambda x \bullet x) \text{ Deriv } 1.) t$

**plus\_deriv\_thm**

$$\begin{aligned}
& \vdash \forall f1 c1 x f2 c2 \\
& \bullet (f1 \text{ Deriv } c1) x \wedge (f2 \text{ Deriv } c2) x \\
& \Rightarrow ((\lambda y \bullet f1 y + f2 y) \text{ Deriv } c1 + c2) x
\end{aligned}$$

**plus\_const\_deriv\_thm**

$$\vdash \forall c x \bullet (\$+ c \text{ Deriv } 1.) x$$

**times\_deriv\_thm**

$$\begin{aligned}
& \vdash \forall f1 c1 x f2 c2 \\
& \bullet (f1 \text{ Deriv } c1) x \wedge (f2 \text{ Deriv } c2) x \\
& \Rightarrow ((\lambda y \bullet f1 y * f2 y) \\
& \quad \text{Deriv } c1 * f2 x + f1 x * c2) \\
& \quad x
\end{aligned}$$

**times\_const\_deriv\_thm**

$$\vdash \forall c x \bullet (\$* c \text{ Deriv } c) x$$

**comp\_deriv\_thm**

$$\begin{aligned}
& \vdash \forall f1 c1 f2 c2 x \\
& \bullet (f1 \text{ Deriv } c1) (f2 x) \wedge (f2 \text{ Deriv } c2) x \\
& \Rightarrow ((\lambda y \bullet f1 (f2 y)) \text{ Deriv } c1 * c2) x
\end{aligned}$$

**minus\_deriv\_thm**

$$\vdash \forall c t \bullet (\sim \text{Deriv } \sim 1.) t$$

**minus\_comp\_deriv\_thm**

$$\vdash \forall f c t \bullet (f \text{ Deriv } c) t \Rightarrow ((\lambda x \bullet \sim (f x)) \text{ Deriv } \sim c) t$$

**$\mathbb{R}_N$ \_exp\_deriv\_thm**

$$\vdash \forall n t$$

$$\bullet ((\lambda x \bullet x \wedge (n + 1)) \text{ Deriv } (\mathbb{N} n + 1.) * t \wedge n) t$$

**recip\_deriv\_thm**

$$\vdash \forall t \bullet \neg t = 0. \Rightarrow (\$^{-1} \text{ Deriv } \sim (t^{-1} * t^{-1})) t$$

**recip\_comp\_deriv\_thm**

$$\vdash \forall f c t$$

$$\bullet (f \text{ Deriv } c) t \wedge \neg f t = 0.$$

$$\Rightarrow ((\lambda x \bullet f x^{-1})$$

$$\text{ Deriv } \sim (f t^{-1} * f t^{-1}) * c)$$

$t$

**poly\_deriv\_thm**

$$\vdash \forall l x$$

$$\bullet (\text{PolyEval } l \text{ Deriv PolyEval } (\text{DerivCoeffs } l) x) x$$

**deriv\_local\_thm**

$$\vdash \forall f g x a b c$$

$$\bullet a < x$$

$$\wedge x < b$$

$$\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow f y = g y)$$

$$\wedge (g \text{ Deriv } c) x$$

$$\Rightarrow (f \text{ Deriv } c) x$$

**deriv\_lim\_fun\_thm**

$$\vdash \forall f c x$$

$$\bullet (f \text{ Deriv } c) x$$

$$\Leftrightarrow ((\lambda y \bullet (f y - f x) / (y - x)) \dashrightarrow c) x$$

**deriv\_unique\_thm**

$$\vdash \forall f x c d \bullet (f \text{ Deriv } c) x \wedge (f \text{ Deriv } d) x \Rightarrow c = d$$

**curtain\_induction\_thm**

$$\vdash \forall p$$

$$\bullet (\exists x \bullet p x)$$

$$\wedge (\forall x \bullet p x \Rightarrow (\forall y \bullet y < x \Rightarrow p y))$$

$$\wedge (\forall x \bullet \exists y z \bullet y < x \wedge x < z \wedge (p y \Rightarrow p z))$$

$$\Rightarrow (\forall x \bullet p x)$$

**bisection\_thm**

$$\vdash \forall p a b$$

$$\bullet (\forall x y z \bullet x \leq y \wedge y \leq z \wedge p x y \wedge p y z \Rightarrow p x z)$$

$$\wedge (\forall y$$

$$\bullet \exists d$$

$$\bullet 0. < d$$

$$\wedge (\forall x z$$

$$\bullet x \leq y \wedge y \leq z \wedge z - x < d \Rightarrow p x z))$$

$$\wedge a \leq b$$

$$\Rightarrow p a b$$

**closed\_interval\_compact\_thm**

$$\vdash \forall x y \bullet \text{ClosedInterval } x y \in \text{Compact}_R$$

**finite\_chain\_thm**

$$\vdash \forall V$$

- $V \in \text{Finite}$   
 $\wedge \neg V = \{\}$   
 $\wedge (\forall A B \bullet A \in V \wedge B \in V \Rightarrow A \subseteq B \vee B \subseteq A)$   
 $\Rightarrow (\exists A \bullet A \in V \wedge \bigcup V = A)$

*cts\_compact\_maximum\_thm*

- $\vdash \forall X f$   
  - $\neg X = \{\} \wedge X \in \text{Compact}_R \wedge (\forall x \bullet f \text{ Cts } x)$   
 $\Rightarrow (\exists x \bullet x \in X \wedge (\forall z \bullet z \in X \Rightarrow f z \leq f x))$

*cts\_maximum\_thm*

- $\vdash \forall f a b$   
  - $a \leq b \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x)$   
 $\Rightarrow (\exists x$   
    - $a \leq x$   
 $\wedge x \leq b$   
 $\wedge (\forall z \bullet a \leq z \wedge z \leq b \Rightarrow f z \leq f x))$

*cts\_abs\_bounded\_thm*

- $\vdash \forall f a b$   
  - $a \leq b \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x)$   
 $\Rightarrow (\exists c \bullet \forall z \bullet a \leq z \wedge z \leq b \Rightarrow \text{Abs } (f z) \leq c)$

*intermediate\_value\_thm*

- $\vdash \forall f a b$   
  - $a < b \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x)$   
 $\Rightarrow (\forall y$   
    - $f a < y \wedge y < f b \vee f b < y \wedge y < f a$   
 $\Rightarrow (\exists x \bullet a < x \wedge x < b \wedge f x = y))$

*local\_min\_thm*

- $\vdash \forall f x a b c$   
  - $a < x$   
 $\wedge x < b$   
 $\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow f x \leq f y)$   
 $\wedge (f \text{ Deriv } c) x$   
 $\Rightarrow c = 0.$

*local\_max\_thm*

- $\vdash \forall f x a b c$   
  - $a < x$   
 $\wedge x < b$   
 $\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow f y \leq f x)$   
 $\wedge (f \text{ Deriv } c) x$   
 $\Rightarrow c = 0.$

*rolle\_thm*

- $\vdash \forall f df a b$   
  - $a < b$   
 $\wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } df x) x)$   
 $\wedge f a = f b$   
 $\Rightarrow (\exists x \bullet a < x \wedge x < b \wedge (f \text{ Deriv } 0.) x)$

*cauchy\_mean\_value\_thm*

- $\vdash \forall f df g dg a b$   
  - $a < b$   
 $\wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } df x) x)$   
 $\wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow g \text{ Cts } x)$

$$\begin{aligned}
& \wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (g \text{ Deriv } dg \ x) \ x) \\
\Rightarrow & (\exists x \\
& \bullet a < x \\
& \wedge x < b \\
& \wedge dg \ x * (f \ b - f \ a) = df \ x * (g \ b - g \ a))
\end{aligned}$$

**mean\_value\_thm**

$$\begin{aligned}
& \vdash \forall f \ df \ a \ b \\
& \bullet a < b \\
& \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \ Cts \ x) \\
& \wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } df \ x) \ x) \\
\Rightarrow & (\exists x \\
& \bullet a < x \wedge x < b \wedge f \ b - f \ a = (b - a) * df \ x)
\end{aligned}$$

**mean\_value\_thm1**

$$\begin{aligned}
& \vdash \forall f \ df \ a \ b \\
& \bullet a < b \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow (f \text{ Deriv } df \ x) \ x) \\
\Rightarrow & (\exists x \\
& \bullet a < x \wedge x < b \wedge f \ b - f \ a = (b - a) * df \ x)
\end{aligned}$$

**deriv\_0\_thm1**

$$\begin{aligned}
& \vdash \forall f \ a \ b \\
& \bullet (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } 0.) \ x) \\
\Rightarrow & (\forall x \ y \bullet a < x \wedge x < y \wedge y < b \Rightarrow f \ x = f \ y)
\end{aligned}$$

**deriv\_0\_thm**

$$\begin{aligned}
& \vdash \forall f \ a \ b \\
& \bullet (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } 0.) \ x) \\
\Rightarrow & (\exists c \bullet \forall x \bullet a < x \wedge x < b \Rightarrow f \ x = c)
\end{aligned}$$

**deriv\_0\_thm2**

$$\vdash \forall f \ x \ y \bullet (\forall x \bullet (f \text{ Deriv } 0.) \ x) \Rightarrow f \ x = f \ y$$

**deriv\_0\_less\_thm**

$$\begin{aligned}
& \vdash \forall f \ df \ a \ b \\
& \bullet a < b \\
& \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow f \ Cts \ x) \\
& \wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } df \ x) \ x) \\
& \wedge (\forall x \bullet a < x \wedge x < b \Rightarrow 0. < df \ x) \\
\Rightarrow & f \ a < f \ b
\end{aligned}$$

**deriv\_linear\_estimate\_thm**

$$\begin{aligned}
& \vdash \forall f \ df \ a \ b \\
& \bullet a < b \\
& \wedge (\forall x \bullet a \leq x \wedge x \leq b \Rightarrow (f \text{ Deriv } df \ x) \ x) \\
& \wedge (\forall x \bullet a < x \wedge x < b \Rightarrow Abs \ (df \ x) \leq 1.) \\
\Rightarrow & Abs \ (f \ b - f \ a) \leq b - a
\end{aligned}$$

**$\mathbb{R}$ \_mono\_inc\_seq\_thm**

$$\begin{aligned}
& \vdash \forall f \\
& \bullet (\forall m \bullet f \ m \leq f \ (m + 1)) \Rightarrow (\forall m \ n \bullet f \ m \leq f \ (m + n))
\end{aligned}$$

**$\mathbb{R}$ \_mono\_dec\_seq\_thm**

$$\begin{aligned}
& \vdash \forall f \\
& \bullet (\forall m \bullet f \ (m + 1) \leq f \ m) \Rightarrow (\forall m \ n \bullet f \ (m + n) \leq f \ m)
\end{aligned}$$

**nested\_interval\_bounds\_thm**

$$\begin{aligned}
& \vdash \forall L \ U \\
& \bullet (\forall m \\
& \bullet L \ m \leq L \ (m + 1) \wedge U \ (m + 1) \leq U \ m \wedge L \ m \leq U \ m) \\
\Rightarrow & (\forall m \ n \bullet L \ m \leq U \ n)
\end{aligned}$$

**nested\_interval\_diag\_thm**

$$\begin{aligned}
& \vdash \forall X \\
& \bullet \exists L \ U
\end{aligned}$$

- $\forall m$ 
  - $L m \leq L (m + 1)$
  - $\wedge U (m + 1) \leq U m$
  - $\wedge L m < U m$
  - $\wedge (X m < L m \vee U m < X m)$

**nested\_interval\_intersection\_thm**

- $\forall L U$ 
  - $(\forall m$ 
    - $L m \leq L (m + 1) \wedge U (m + 1) \leq U m \wedge L m \leq U m)$
  - $\Rightarrow (\exists x \bullet \forall m \bullet L m \leq x \wedge x \leq U m)$

**$\mathbb{R}$ \_uncountable\_thm**

- $\forall X \bullet \exists x \bullet \forall m \bullet \neg x = X m$

**$\cap$ \_open\_interval\_thm**

- $\forall x1 y1 x2 y2$ 
  - $\exists x y$
  - $OpenInterval x1 y1 \cap OpenInterval x2 y2 = OpenInterval x y$

**$\cap$ \_closed\_interval\_thm**

- $\forall x1 y1 x2 y2$ 
  - $\exists x y$
  - $ClosedInterval x1 y1 \cap ClosedInterval x2 y2 = ClosedInterval x y$

**$\cup$ \_open\_ $\mathbb{R}$ \_thm**  $\vdash \forall V \bullet V \subseteq Open_R \Rightarrow \bigcup V \in Open_R$

**$\cup$ \_open\_ $\mathbb{R}$ \_thm**

- $\forall X Y \bullet X \in Open_R \wedge Y \in Open_R \Rightarrow X \cup Y \in Open_R$

**$\cap$ \_open\_ $\mathbb{R}$ \_thm**

- $\forall X Y \bullet X \in Open_R \wedge Y \in Open_R \Rightarrow X \cap Y \in Open_R$

**$\cap$ \_closed\_ $\mathbb{R}$ \_thm**

- $\forall V \bullet V \subseteq Closed_R \Rightarrow \bigcap V \in Closed_R$

**$\cap$ \_closed\_ $\mathbb{R}$ \_thm**

- $\forall X Y \bullet X \in Closed_R \wedge Y \in Closed_R \Rightarrow X \cap Y \in Closed_R$

**$\cup$ \_closed\_ $\mathbb{R}$ \_thm**

- $\forall X Y \bullet X \in Closed_R \wedge Y \in Closed_R \Rightarrow X \cup Y \in Closed_R$

**open\_interval\_open\_thm**

- $\forall x y \bullet OpenInterval x y \in Open_R$

**complement\_open\_interval\_thm**

- $\forall x y \bullet \sim (OpenInterval x y) = \{t | t \leq x\} \cup \{t | y \leq t\}$

**complement\_closed\_interval\_thm**

- $\forall x y \bullet \sim (ClosedInterval x y) = \{t | t < x\} \cup \{t | y < t\}$

**half\_infinite\_intervals\_open\_thm**

- $\forall x \bullet \{t | t < x\} \in Open_R \wedge \{t | x < t\} \in Open_R$

**half\_infinite\_intervals\_closed\_thm**

- $\forall x \bullet \{t | t \leq x\} \in Closed_R \wedge \{t | x \leq t\} \in Closed_R$

**empty\_universe\_open\_closed\_thm**

- $\{\} \in Open_R$
- $\wedge Universe \in Open_R$
- $\{\} \in Closed_R$
- $\wedge Universe \in Closed_R$

**compact\_closed\_thm**

- $Compact_R \subseteq Closed_R$

**compact\_min\_max\_thm**



- $$\vdash \forall X$$
- $\neg X = \{\} \wedge X \in \text{Compact}_R$
  - $\Rightarrow (\exists L U$
  - $L \in X \wedge U \in X \wedge (\forall x \bullet x \in X \Rightarrow L \leq x \wedge x \leq U))$

***closed\_⊆\_compact\_thm***

- $$\vdash \forall X Y$$
- $Y \in \text{Closed}_R \wedge Y \subseteq X \wedge X \in \text{Compact}_R$
  - $\Rightarrow Y \in \text{Compact}_R$

***empty\_universe\_compact\_thm***

- $$\vdash \{\} \in \text{Compact}_R \wedge \neg \text{Universe} \in \text{Compact}_R$$

***heine\_borel\_thm***

- $$\vdash \forall X$$
- $X \in \text{Compact}_R$
  - $\Leftrightarrow X \in \text{Closed}_R$
  - $\wedge (\exists L U \bullet \forall x \bullet x \in X \Rightarrow L \leq x \wedge x \leq U)$

***ℝ\_connected\_thm***

- $$\vdash \forall X Y$$
- $\neg X = \{\}$
  - $\wedge X \in \text{Open}_R$
  - $\wedge \neg Y = \{\}$
  - $\wedge Y \in \text{Open}_R$
  - $\wedge (\forall x \bullet x \in X \cup Y)$
  - $\Rightarrow (\exists x \bullet x \in X \cap Y)$

***lim\_seq\_cauchy\_seq\_thm***

- $$\vdash \forall s x$$
- $s \rightarrow x$
  - $\Rightarrow (\forall e$
  - $0. < e$
  - $\Rightarrow (\exists n$
  - $\forall k m$
  - $n \leq k \wedge n \leq m \Rightarrow \text{Abs } (s k - s m) < e)$

***fin\_seq\_bounded\_thm***

- $$\vdash \forall s n \bullet \exists b \bullet \forall m \bullet m \leq n \Rightarrow s m < b$$

***cauchy\_seq\_bounded\_above\_thm***

- $$\vdash \forall s$$
- $(\forall e$
  - $0. < e$
  - $\Rightarrow (\exists n$
  - $\forall k m$
  - $n \leq k \wedge n \leq m \Rightarrow \text{Abs } (s k - s m) < e)$
  - $\Rightarrow (\exists b \bullet \forall m \bullet s m < b)$

***cauchy\_seq\_bounded\_below\_thm***

- $$\vdash \forall s$$
- $(\forall e$
  - $0. < e$
  - $\Rightarrow (\exists n$
  - $\forall k m$
  - $n \leq k \wedge n \leq m \Rightarrow \text{Abs } (s k - s m) < e)$
  - $\Rightarrow (\exists b \bullet \forall m \bullet b < s m)$

***lim\_seq\_mono\_inc\_sup\_thm***

- $$\vdash \forall s ub$$

- $(\forall m \bullet s\ m \leq ub \wedge s\ m \leq s\ (m + 1))$   
 $\Rightarrow s \rightarrow Sup\ \{t \mid \exists m \bullet t = s\ m\}$   
 $\wedge (\forall m \bullet s\ m \leq Sup\ \{t \mid \exists m \bullet t = s\ m\})$

**lim\_seq\_mono\_inc\_thm**

- $\vdash \forall s\ ub$
- $(\forall m \bullet s\ m \leq ub \wedge s\ m \leq s\ (m + 1))$   
 $\Rightarrow (\exists x \bullet s \rightarrow x \wedge (\forall m \bullet s\ m \leq x))$

**lim\_seq\_mono\_dec\_thm**

- $\vdash \forall s\ lb$
- $(\forall m \bullet lb \leq s\ m \wedge s\ (m + 1) \leq s\ m)$   
 $\Rightarrow (\exists x \bullet s \rightarrow x \wedge (\forall m \bullet x \leq s\ m))$

**lim\_sup\_thm**

- $\vdash \forall s\ lb\ ub$
- $(\forall m \bullet lb \leq s\ m \wedge s\ m \leq ub)$   
 $\Rightarrow (\exists\ lim\_sup$
- $\forall e$
- $0. < e$   
 $\Rightarrow (\exists n \bullet \forall m \bullet n \leq m \Rightarrow s\ m < lim\_sup + e)$   
 $\wedge (\forall n \bullet \exists m \bullet n \leq m \wedge lim\_sup < s\ m + e))$

**lim\_inf\_thm**

- $\vdash \forall s\ lb\ ub$
- $(\forall m \bullet lb \leq s\ m \wedge s\ m \leq ub)$   
 $\Rightarrow (\exists\ lim\_inf$
- $\forall e$
- $0. < e$   
 $\Rightarrow (\exists n \bullet \forall m \bullet n \leq m \Rightarrow lim\_inf - e < s\ m)$   
 $\wedge (\forall n \bullet \exists m \bullet n \leq m \wedge s\ m - e < lim\_inf))$

**cauchy\_seq\_lim\_seq\_thm**

- $\vdash \forall s$
- $(\forall e$
- $0. < e$   
 $\Rightarrow (\exists n$
- $\forall k\ m$   
 $\bullet n \leq k \wedge n \leq m \Rightarrow Abs\ (s\ k - s\ m) < e)$
- $\Rightarrow (\exists x \bullet s \rightarrow x)$

**lim\_seq\_iff\_cauchy\_seq\_thm**

- $\vdash \forall s$
- $(\exists x \bullet s \rightarrow x)$   
 $\Leftrightarrow (\forall e$
- $0. < e$   
 $\Rightarrow (\exists n$
- $\forall k\ m$   
 $\bullet n \leq k \wedge n \leq m \Rightarrow Abs\ (s\ k - s\ m) < e)$

**lim\_fun\_lim\_seq\_thm**

- $\vdash \forall f\ c\ x$
- $(f \dashrightarrow c)\ x$   
 $\Leftrightarrow (\forall s$
- $s \rightarrow x \wedge (\forall m \bullet \neg s\ m = x)$   
 $\Rightarrow (\lambda m \bullet f\ (s\ m)) \rightarrow c)$

**const\_lim\_fun\_thm**

- $\vdash \forall c\ t \bullet ((\lambda x \bullet c) \dashrightarrow c)\ t$

**id\_lim\_fun\_thm**

- $\vdash \forall t \bullet ((\lambda x \bullet x) \dashrightarrow t)\ t$

**plus\_lim\_fun\_thm**

$$\begin{aligned} &\vdash \forall f1\ t1\ x\ f2\ t2 \\ &\bullet (f1 \dashrightarrow t1)\ x \wedge (f2 \dashrightarrow t2)\ x \\ &\Rightarrow ((\lambda x \bullet f1\ x + f2\ x) \dashrightarrow t1 + t2)\ x \end{aligned}$$

**times\_lim\_fun\_thm**

$$\begin{aligned} &\vdash \forall f1\ t1\ x\ f2\ t2 \\ &\bullet (f1 \dashrightarrow t1)\ x \wedge (f2 \dashrightarrow t2)\ x \\ &\Rightarrow ((\lambda x \bullet f1\ x * f2\ x) \dashrightarrow t1 * t2)\ x \end{aligned}$$

**recip\_lim\_fun\_thm**

$$\begin{aligned} &\vdash \forall f\ t\ x \\ &\bullet (f \dashrightarrow t)\ x \wedge \neg t = 0. \\ &\Rightarrow ((\lambda x \bullet f\ x^{-1}) \dashrightarrow t^{-1})\ x \end{aligned}$$

**comp\_lim\_fun\_thm**

$$\begin{aligned} &\vdash \forall f\ t\ x\ g \\ &\bullet (f \dashrightarrow t)\ x \wedge g\ Cts\ t \Rightarrow ((\lambda x \bullet g\ (f\ x)) \dashrightarrow g\ t)\ x \end{aligned}$$

**cts\_lim\_fun\_thm**

$$\vdash \forall f\ x \bullet f\ Cts\ x \Rightarrow (f \dashrightarrow f\ x)\ x$$

**comp\_lim\_fun\_thm1**

$$\begin{aligned} &\vdash \forall f\ t\ x\ g \\ &\bullet (g \dashrightarrow t)\ (f\ x) \\ &\quad \wedge f\ Cts\ x \\ &\quad \wedge (\forall y \bullet f\ y = f\ x \Leftrightarrow y = x) \\ &\Rightarrow ((\lambda x \bullet g\ (f\ x)) \dashrightarrow t)\ x \end{aligned}$$

**poly\_lim\_fun\_thm**

$$\vdash \forall f\ x \bullet f \in PolyFunc \Rightarrow (f \dashrightarrow f\ x)\ x$$

**lim\_fun\_upper\_bound\_thm**

$$\begin{aligned} &\vdash \forall u\ d\ c\ t\ x \\ &\bullet 0. < d \\ &\quad \wedge (\forall y \bullet Abs\ (y - x) < d \wedge \neg y = x \Rightarrow u\ y \leq c) \\ &\quad \wedge (u \dashrightarrow t)\ x \\ &\Rightarrow t \leq c \end{aligned}$$

**lim\_fun\_unique\_thm**

$$\vdash \forall u\ s\ t\ x \bullet (u \dashrightarrow s)\ x \wedge (u \dashrightarrow t)\ x \Rightarrow s = t$$

**lim\_fun\_local\_thm**

$$\begin{aligned} &\vdash \forall u\ v\ s\ t\ x\ a\ b \\ &\bullet (u \dashrightarrow s)\ x \\ &\quad \wedge a < x \\ &\quad \wedge x < b \\ &\quad \wedge (\forall t \bullet a < t \wedge t < b \wedge \neg t = x \Rightarrow u\ t = v\ t) \\ &\Rightarrow (v \dashrightarrow s)\ x \end{aligned}$$

**deriv\_lim\_fun\_thm1**

$$\begin{aligned} &\vdash \forall f\ c\ x \\ &\bullet (f\ Deriv\ c)\ x \\ &\Leftrightarrow ((\lambda y \bullet (f\ y - f\ x) * (y - x)^{-1}) \dashrightarrow c)\ x \end{aligned}$$

**comp\_lim\_fun\_thm2**

$$\begin{aligned} &\vdash \forall f\ a\ b\ t\ x\ g \\ &\bullet x \in OpenInterval\ a\ b \\ &\quad \wedge (g \dashrightarrow t)\ (f\ x) \\ &\quad \wedge f\ Cts\ x \\ &\quad \wedge (\forall y \\ &\quad \bullet y \in OpenInterval\ a\ b \Rightarrow (f\ y = f\ x \Leftrightarrow y = x)) \end{aligned}$$

$$\Rightarrow ((\lambda x \bullet g (f x)) \dashrightarrow t) x$$

**inverse\_deriv\_thm**

$$\begin{aligned} & \vdash \forall f g a b x c \\ & \bullet x \in \text{OpenInterval } a b \\ & \quad \wedge (\forall y \bullet y \in \text{OpenInterval } a b \Rightarrow f (g y) = y) \\ & \quad \wedge (f \text{ Deriv } c) (g x) \\ & \quad \wedge g \text{ Cts } x \\ & \quad \wedge \neg c = 0. \\ & \Rightarrow (g \text{ Deriv } c^{-1}) x \end{aligned}$$

**const\_unif\_lim\_seq\_thm**

$$\vdash \forall h X \bullet ((\lambda m \bullet h) \dashrightarrow h) X$$

**plus\_unif\_lim\_seq\_thm**

$$\begin{aligned} & \vdash \forall u1 h1 X u2 h2 \\ & \bullet (u1 \dashrightarrow h1) X \wedge (u2 \dashrightarrow h2) X \\ & \Rightarrow ((\lambda m y \bullet u1 m y + u2 m y) \\ & \quad \dashrightarrow (\lambda y \bullet h1 y + h2 y)) \\ & \quad X \end{aligned}$$

**bounded\_unif\_limit\_thm**

$$\begin{aligned} & \vdash \forall u g X c \\ & \bullet (u \dashrightarrow g) X \wedge (\forall y m \bullet y \in X \Rightarrow \text{Abs } (u m y) < c) \\ & \Rightarrow (\exists d \bullet \forall y \bullet y \in X \Rightarrow \text{Abs } (g y) < d) \end{aligned}$$

**times\_unif\_lim\_seq\_thm**

$$\begin{aligned} & \vdash \forall u g v h X c d \\ & \bullet (u \dashrightarrow g) X \\ & \quad \wedge (v \dashrightarrow h) X \\ & \quad \wedge (\forall y m \bullet y \in X \Rightarrow \text{Abs } (u m y) < c) \\ & \quad \wedge (\forall y m \bullet y \in X \Rightarrow \text{Abs } (v m y) < d) \\ & \Rightarrow ((\lambda m y \bullet u m y * v m y) \dashrightarrow (\lambda y \bullet g y * h y)) \\ & \quad X \end{aligned}$$

**unif\_lim\_seq\_bounded\_thm**

$$\begin{aligned} & \vdash \forall u h a b \\ & \bullet a < b \\ & \quad \wedge (u \dashrightarrow h) (\text{ClosedInterval } a b) \\ & \quad \wedge (\forall y \bullet a \leq y \wedge y \leq b \Rightarrow h \text{ Cts } y) \\ & \Rightarrow (\exists c n \\ & \bullet \forall m y \bullet n \leq m \wedge a \leq y \wedge y \leq b \Rightarrow \text{Abs } (u m y) < c) \end{aligned}$$

**unif\_lim\_seq\_subset\_thm**

$$\vdash \forall u h X Y \bullet (u \dashrightarrow h) X \wedge Y \subseteq X \Rightarrow (u \dashrightarrow h) Y$$

**unif\_lim\_seq\_shift\_thm**

$$\vdash \forall m u h X \bullet (u \dashrightarrow h) X \Leftrightarrow ((\lambda n \bullet u (n + m)) \dashrightarrow h) X$$

**unif\_lim\_seq\_cts\_thm**

$$\begin{aligned} & \vdash \forall u h x a b \\ & \bullet (u \dashrightarrow h) (\text{OpenInterval } a b) \\ & \quad \wedge a < x \\ & \quad \wedge x < b \\ & \quad \wedge (\forall m \bullet u m \text{ Cts } x) \\ & \Rightarrow h \text{ Cts } x \end{aligned}$$

**unif\_lim\_seq\_cauchy\_seq\_thm**

$$\begin{aligned} & \vdash \forall u f X \\ & \bullet (u \dashrightarrow f) X \\ & \Rightarrow (\forall e \end{aligned}$$

- $0. < e$
- $\Rightarrow (\exists n$
- $\forall k m$
- $n \leq k \wedge n \leq m$
- $\Rightarrow (\forall y$
- $y \in X \Rightarrow \text{Abs } (u k y - u m y) < e))$

**cauchy\_seq\_unif\_lim\_seq\_thm**

- $\vdash \forall u X$
- $(\forall e$
- $0. < e$
- $\Rightarrow (\exists n$
- $\forall k m$
- $n \leq k \wedge n \leq m$
- $\Rightarrow (\forall y$
- $y \in X \Rightarrow \text{Abs } (u k y - u m y) < e))$
- $\Rightarrow (\exists f \bullet (u \dashrightarrow f) X)$

**unif\_lim\_seq\_pointwise\_lim\_seq\_thm**

- $\vdash \forall u f X x \bullet (u \dashrightarrow f) X \wedge x \in X \Rightarrow (\lambda m \bullet u m x) \rightarrow f x$

**unif\_lim\_seq\_pointwise\_unique\_thm**

- $\vdash \forall u f g X x c$
- $(u \dashrightarrow f) X \wedge x \in X \wedge (\lambda m \bullet u m x) \rightarrow c \Rightarrow f x = c$

**unif\_lim\_seq\_unique\_thm**

- $\vdash \forall u f g X x$
- $(u \dashrightarrow f) X \wedge (u \dashrightarrow g) X \wedge x \in X \Rightarrow f x = g x$

**lim\_fun\_lim\_seq\_interchange\_thm**

- $\vdash \forall u f a b x s$
- $(u \dashrightarrow f) (\text{OpenInterval } a b \setminus \{x\})$
- $\wedge a < x$
- $\wedge x < b$
- $\wedge (\forall m \bullet (u m \dashrightarrow s m) x)$
- $\Rightarrow (\exists c \bullet (f \dashrightarrow c) x \wedge s \rightarrow c)$

**unif\_lim\_seq\_deriv\_estimate\_thm**

- $\vdash \forall du df x0 y0 A B u e$
- $(du \dashrightarrow df) (\text{OpenInterval } A B)$
- $\wedge (\forall y m \bullet A < y \wedge y < B \Rightarrow (u m \text{ Deriv } du m y) y)$
- $\wedge A < x0$
- $\wedge x0 < B$
- $\wedge (\lambda m \bullet u m x0) \rightarrow y0$
- $\wedge 0. < e$
- $\Rightarrow (\exists N$
- $\forall n m x t$
- $N \leq n \wedge N \leq m \wedge A < x \wedge x < B \wedge A < t \wedge t < B$
- $\Rightarrow \text{Abs}$
- $(u n t + \sim (u m t) + \sim (u n x) + u m x)$
- $\leq \text{Abs } (t + \sim x) * e)$

**unif\_lim\_seq\_deriv\_thm**

- $\vdash \forall u du df A B x0 y0$
- $(du \dashrightarrow df) (\text{OpenInterval } A B)$
- $\wedge (\forall y m$
- $y \in \text{OpenInterval } A B \Rightarrow (u m \text{ Deriv } du m y) y)$
- $\wedge x0 \in \text{OpenInterval } A B$

$$\begin{aligned}
& \wedge (\lambda m \bullet u \ m \ x0) \rightarrow y0 \\
& \Rightarrow (\exists f \\
& \bullet (u \dashrightarrow f) \text{ (OpenInterval A B)} \\
& \wedge (\forall y \\
& \bullet y \in \text{OpenInterval A B} \Rightarrow (f \text{ Deriv df } y) y))
\end{aligned}$$

**power\_series\_series\_thm**

$$\begin{aligned}
& \vdash \forall s \\
& \bullet \text{PowerSeries } s = (\lambda n \bullet \text{Sigma } (\lambda m \bullet s \ m * x \wedge m) \ n)
\end{aligned}$$

**series\_power\_series\_thm**

$$\vdash \forall s \bullet \text{Sigma } s = (\lambda n \bullet \text{PowerSeries } s \ n \ 1.)$$

**power\_series\_0\_arg\_thm**

$$\begin{aligned}
& \vdash (\forall s \bullet \text{PowerSeries } s \ 0 \ 0. = 0.) \\
& \wedge (\forall s \bullet \text{PowerSeries } s \ (m + 1) \ 0. = s \ 0)
\end{aligned}$$

**power\_series\_limit\_0\_thm**

$$\vdash \forall s \bullet (\lambda m \bullet \text{PowerSeries } s \ m \ 0.) \rightarrow s \ 0$$

**plus\_series\_thm**

$$\begin{aligned}
& \vdash \forall s1 \ s2 \\
& \bullet \text{Sigma } (\lambda m \bullet s1 \ m + s2 \ m) \\
& = (\lambda n \bullet \text{Sigma } s1 \ n + \text{Sigma } s2 \ n)
\end{aligned}$$

**series\_0\_thm**  $\vdash \text{Sigma } (\lambda m \bullet 0.) = (\lambda m \bullet 0.)$

**const\_times\_series\_thm**

$$\vdash \forall s \ c \bullet \text{Sigma } (\lambda m \bullet c * s \ m) = (\lambda n \bullet c * \text{Sigma } s \ n)$$

**lim\_seq\_NN\_recip\_eq\_0\_thm**

$$\vdash (\lambda m \bullet \text{NR } m^{-1}) \rightarrow 0.$$

**lim\_seq\_RN\_exp\_eq\_0\_thm**

$$\vdash \forall x \bullet \sim 1. < x \wedge x < 1. \Rightarrow (\lambda m \bullet x \wedge m) \rightarrow 0.$$

**series\_shift\_thm**

$$\begin{aligned}
& \vdash \forall n \ s \\
& \bullet \text{Sigma } (\lambda m \bullet s \ (m + n)) \\
& = (\lambda m \bullet \text{Sigma } s \ (m + n) - \text{Sigma } s \ n)
\end{aligned}$$

**lim\_series\_shift\_thm**

$$\begin{aligned}
& \vdash \forall n \ s \ x \\
& \bullet \text{Sigma } s \rightarrow x \\
& \Leftrightarrow \text{Sigma } (\lambda m \bullet s \ (m + n)) \rightarrow x - \text{Sigma } s \ n
\end{aligned}$$

**lim\_series\_shift\_Existence\_thm**

$$\begin{aligned}
& \vdash \forall n \ s \\
& \bullet (\exists x \bullet \text{Sigma } s \rightarrow x) \\
& \Leftrightarrow (\exists x \bullet \text{Sigma } (\lambda m \bullet s \ (m + n)) \rightarrow x)
\end{aligned}$$

**lim\_series\_bounded\_thm**

$$\begin{aligned}
& \vdash \forall e \ s \ c \\
& \bullet 0. < e \wedge \text{Sigma } s \rightarrow c \\
& \Rightarrow (\exists m \bullet \forall n \bullet m \leq n \Rightarrow \text{Abs } (s \ n) < e)
\end{aligned}$$

**geometric\_sum\_thm1**

$$\begin{aligned}
& \vdash \forall n \ x \\
& \bullet \neg x = 1. \\
& \Rightarrow \text{PolyEval } ((\lambda m \bullet 1.) \text{ To } (n + 1)) \ x \\
& = (1. - x \wedge (n + 1)) / (1. - x)
\end{aligned}$$

**geometric\_sum\_thm**

$$\begin{aligned}
& \vdash \forall n \ x \\
& \bullet \neg x = 1. \\
& \Rightarrow \text{Sigma } (\lambda m \bullet x \wedge m) \ (n + 1)
\end{aligned}$$

$$= (1. - x ^ (n + 1)) / (1. - x)$$

**geometric\_series\_thm**

$\vdash \forall x$

- $\sim 1. < x \wedge x < 1.$
- $\Rightarrow (\lambda n \bullet \text{PowerSeries } (\lambda m \bullet 1.) n x)$
- $\rightarrow 1. / (1. - x)$

**geometric\_series\_series\_thm**

$\vdash \forall x$

- $(\lambda n \bullet \text{PowerSeries } (\lambda m \bullet 1.) n x)$
- $= \text{Sigma } (\lambda m \bullet x ^ m)$

**geometric\_series\_thm1**

$\vdash \forall x$

- $\sim 1. < x \wedge x < 1.$
- $\Rightarrow \text{Sigma } (\lambda m \bullet x ^ m) \rightarrow 1. / (1. - x)$

**weierstrass\_test\_thm**

$\vdash \forall u X s c$

- $(\forall m x \bullet x \in X \Rightarrow \text{Abs } (u m x) \leq s m) \wedge \text{Sigma } s \rightarrow c$
- $\Rightarrow (\exists f \bullet ((\lambda m x \bullet \text{Sigma } (\lambda m \bullet u m x) m) \dashrightarrow f) X)$

**comparison\_test\_thm**

$\vdash \forall s1 s2 c2$

- $(\forall m \bullet \text{Abs } (s1 m) \leq s2 m) \wedge \text{Sigma } s2 \rightarrow c2$
- $\Rightarrow (\exists c1 \bullet \text{Sigma } s1 \rightarrow c1)$

**simple\_root\_test\_thm**

$\vdash \forall s b m$

- $0. < b \wedge b < 1. \wedge (\forall n \bullet m \leq n \Rightarrow \text{Abs } (s n) \leq b ^ n)$
- $\Rightarrow (\exists c \bullet \text{Sigma } s \rightarrow c)$

**root\_test\_thm**

$\vdash \forall s b d m$

- $b < 1. \wedge (\forall n \bullet m \leq n \Rightarrow \text{Abs } (s n) \leq d * b ^ n)$
- $\Rightarrow (\exists c \bullet \text{Sigma } s \rightarrow c)$

**ratio\_test\_thm1**

$\vdash \forall s b m$

- $(\forall m \bullet \neg s m = 0.)$
- $\wedge 0. < b$
- $\wedge b < 1.$
- $\wedge (\forall n \bullet m \leq n \Rightarrow \text{Abs } (s (n + 1) / s n) < b)$
- $\Rightarrow (\exists c \bullet \text{Sigma } s \rightarrow c)$

**ratio\_test\_thm**

$\vdash \forall s b$

- $(\forall m \bullet \neg s m = 0.)$
- $\wedge 0. \leq b$
- $\wedge b < 1.$
- $\wedge (\lambda n \bullet \text{Abs } (s (n + 1) / s n)) \rightarrow b$
- $\Rightarrow (\exists c \bullet \text{Sigma } s \rightarrow c)$

**power\_series\_convergence\_thm**

$\vdash \forall s B C b$

- $0. < B$
- $\wedge 0. < b$
- $\wedge b < B$
- $\wedge (\lambda m \bullet \text{PowerSeries } (\lambda m \bullet \text{Abs } (s m)) m B) \rightarrow C$
- $\Rightarrow (\exists f$

• (*PowerSeries*  $s \dashrightarrow f$ ) (*OpenInterval* ( $\sim b$ )  $b$ )  
***power\_series\_deriv\_coefs\_thm***

$\vdash \forall s \ n \ x$   
 • (*PowerSeries*  $s \ (n + 1)$ )  
*Deriv PowerSeries*  
 $(\lambda m \bullet \mathbb{NIR} \ (m + 1) * s \ (m + 1))$   
 $n$   
 $x$   
 $x$

***power\_series\_deriv\_lemma1***

$\vdash \forall b$   
 •  $0. < b \wedge b < 1.$   
 $\Rightarrow (\exists c \bullet \text{Sigma} \ (\lambda m \bullet \mathbb{NIR} \ (m + 1) * b \wedge m) \rightarrow c)$

***power\_series\_deriv\_lemma2***

$\vdash \forall s \ C \ b$   
 •  $0. < b$   
 $\wedge b < 1.$   
 $\wedge (\lambda m \bullet \text{PowerSeries} \ (\lambda m \bullet \text{Abs} \ (s \ m)) \ m \ 1.) \rightarrow C$   
 $\Rightarrow (\exists D$   
 •  $(\lambda m$   
 • *PowerSeries*  
 $(\lambda m \bullet \text{Abs} \ (\mathbb{NIR} \ (m + 1) * s \ (m + 1)))$   
 $m$   
 $b$   
 $\rightarrow D)$

***power\_series\_scale\_arg\_thm***

$\vdash \forall c \ s \ m \ y$   
 • *PowerSeries*  $s \ m \ (c * y)$   
 $= \text{PowerSeries} \ (\lambda m \bullet s \ m * c \wedge m) \ m \ y$

***power\_series\_scale\_coefs\_thm***

$\vdash \forall c \ s \ m \ y$   
 • *PowerSeries*  $(\lambda m \bullet c * s \ m) \ m \ y$   
 $= c * \text{PowerSeries} \ s \ m \ y$

***power\_series\_deriv\_lim\_seq\_convergence\_thm***

$\vdash \forall s \ B \ C \ b$   
 •  $0. < b$   
 $\wedge b < B$   
 $\wedge (\lambda m \bullet \text{PowerSeries} \ (\lambda m \bullet \text{Abs} \ (s \ m)) \ m \ B) \rightarrow C$   
 $\Rightarrow (\exists D$   
 •  $(\lambda m$   
 • *PowerSeries*  
 $(\lambda m \bullet \text{Abs} \ (\mathbb{NIR} \ (m + 1) * s \ (m + 1)))$   
 $m$   
 $b$   
 $\rightarrow D)$

***power\_series\_deriv\_convergence\_thm***

$\vdash \forall s \ B \ C \ b$   
 •  $0. < B$   
 $\wedge 0. < b$   
 $\wedge b < B$   
 $\wedge (\lambda m \bullet \text{PowerSeries} \ (\lambda m \bullet \text{Abs} \ (s \ m)) \ m \ B) \rightarrow C$



$$\Rightarrow (\exists df$$

- $(PowerSeries (\lambda m \bullet \mathbb{N}\mathbb{R} (m + 1) * s (m + 1))$   
 $----> df)$   
 $(OpenInterval (\sim b) b))$

**power\_series\_main\_thm**

$$\vdash \forall s B C b$$

- $0. < b$   
 $\wedge b < B$   
 $\wedge (\lambda m \bullet PowerSeries (\lambda m \bullet Abs (s m)) m B) -> C$

$$\Rightarrow (\exists f df$$

- $(PowerSeries s ----> f) (OpenInterval (\sim b) b)$   
 $\wedge (PowerSeries (\lambda m \bullet \mathbb{N}\mathbb{R} (m + 1) * s (m + 1))$   
 $----> df)$   
 $(OpenInterval (\sim b) b)$   
 $\wedge (\forall y$ 
  - $y \in OpenInterval (\sim b) b$   
 $\Rightarrow (f Deriv df y) y))$

**power\_series\_main\_thm1**

$$\vdash \forall s b$$

- $0. < b$   
 $\wedge (\forall x$ 
  - $0. < x$   
 $\Rightarrow (\exists c$ 
    - $(\lambda m \bullet PowerSeries (\lambda m \bullet Abs (s m)) m x)$   
 $-> c))$

$$\Rightarrow (\exists f df$$

- $(PowerSeries s ----> f) (OpenInterval (\sim b) b)$   
 $\wedge (PowerSeries (\lambda m \bullet \mathbb{N}\mathbb{R} (m + 1) * s (m + 1))$   
 $----> df)$   
 $(OpenInterval (\sim b) b)$   
 $\wedge (\forall y$ 
  - $y \in OpenInterval (\sim b) b$   
 $\Rightarrow (f Deriv df y) y))$

**power\_series\_main\_thm2**

$$\vdash \forall s$$

- $(\forall x$ 
  - $0. < x$   
 $\Rightarrow (\exists c$ 
    - $(\lambda m \bullet PowerSeries (\lambda m \bullet Abs (s m)) m x)$   
 $-> c))$

$$\Rightarrow (\exists f df$$

- $(\forall x \bullet (\lambda m \bullet PowerSeries s m x) -> f x)$   
 $\wedge (\forall x$ 
  - $(\lambda m$ 
    - $PowerSeries$   
 $(\lambda m \bullet \mathbb{N}\mathbb{R} (m + 1) * s (m + 1))$   
 $m$   
 $x)$   
 $-> df x)$

$$\wedge (\forall x \bullet (f Deriv df x) x)$$

$$\wedge f 0. = s 0)$$

**power\_series\_main\_thm3**

$$\begin{aligned}
& \vdash \forall s \\
& \bullet (\forall x \\
& \quad \bullet 0. < x \\
& \quad \Rightarrow (\exists c \\
& \quad \bullet (\lambda m \bullet \text{PowerSeries } (\lambda m \bullet \text{Abs } (s \ m)) \ m \ x) \\
& \quad \quad \rightarrow c)) \\
& \Rightarrow (\exists f \ d1f \ d2f \\
& \bullet (\forall x \bullet (\lambda m \bullet \text{PowerSeries } s \ m \ x) \rightarrow f \ x) \\
& \quad \wedge (\forall x \\
& \quad \bullet (\lambda m \\
& \quad \quad \bullet \text{PowerSeries} \\
& \quad \quad \quad (\lambda m \bullet \text{NR } (m + 1) * s \ (m + 1)) \\
& \quad \quad \quad \ m \\
& \quad \quad \quad \ x) \\
& \quad \quad \rightarrow d1f \ x) \\
& \quad \wedge (\forall x \\
& \quad \bullet (\lambda m \\
& \quad \quad \bullet \text{PowerSeries} \\
& \quad \quad \quad (\lambda m \\
& \quad \quad \quad \bullet \text{NR } (m + 1) \\
& \quad \quad \quad \quad * \text{NR } (m + 2) \\
& \quad \quad \quad \quad * s \ (m + 2)) \\
& \quad \quad \quad \ m \\
& \quad \quad \quad \ x) \\
& \quad \quad \rightarrow d2f \ x) \\
& \quad \wedge (\forall x \bullet (f \ \text{Deriv } d1f \ x) \ x) \\
& \quad \wedge (\forall x \bullet (d1f \ \text{Deriv } d2f \ x) \ x) \\
& \quad \wedge f \ 0. = s \ 0 \\
& \quad \wedge d1f \ 0. = s \ 1)
\end{aligned}$$
**factorial\_linear\_estimate\_thm**

$$\vdash \forall m \bullet 1. \leq \text{NR } (m !) \wedge \text{NR } m \leq \text{NR } (m !)$$
**factorial\_0\_less\_thm**

$$\vdash \forall m \bullet 0. < \text{NR } (m !)$$
**factorial\_times\_recip\_thm**

$$\vdash \forall m \bullet \text{NR } (m + 1) * \text{NR } ((m + 1) !) ^{-1} = \text{NR } (m !) ^{-1}$$
**gen\_rolle\_thm**

$$\begin{aligned}
& \vdash \forall n \ D \ a \ b \\
& \bullet a < b \\
& \quad \wedge (\forall m \ x \bullet m \leq n \wedge a \leq x \wedge x \leq b \Rightarrow D \ m \ \text{Cts } x) \\
& \quad \wedge (\forall m \ x \\
& \quad \bullet m \leq n \wedge a < x \wedge x < b \\
& \quad \quad \Rightarrow (D \ m \ \text{Deriv } D \ (m + 1) \ x) \ x) \\
& \quad \wedge (\forall m \bullet m \leq n \Rightarrow D \ m \ a = 0.) \\
& \quad \wedge D \ 0 \ b = 0. \\
& \Rightarrow (\exists x \bullet a < x \wedge x < b \wedge D \ (n + 1) \ x = 0.)
\end{aligned}$$
**taylor\_thm**

$$\begin{aligned}
& \vdash \forall n \ f \ D \ a \ b \\
& \bullet a < b \\
& \quad \wedge D \ 0 = f \\
& \quad \wedge (\forall m \ x \bullet m \leq n \wedge a \leq x \wedge x \leq b \Rightarrow D \ m \ \text{Cts } x) \\
& \quad \wedge (\forall m \ x
\end{aligned}$$

$$\begin{aligned}
& \bullet m \leq n \wedge a < x \wedge x < b \\
& \Rightarrow (D \ m \ Deriv \ D \ (m + 1) \ x) \ x) \\
\Rightarrow & (\exists \ x \\
& \bullet a < x \\
& \quad \wedge x < b \\
& \quad \wedge f \ b \\
& \quad = \text{PowerSeries} \\
& \quad \quad (\lambda \ m \bullet D \ m \ a \ * \ \mathbb{N}\mathbb{R} \ (m \ !)^{-1}) \\
& \quad \quad \quad (n + 1) \\
& \quad \quad \quad (b - a) \\
& \quad \quad + D \ (n + 1) \ x \\
& \quad \quad \quad * (b - a) \wedge (n + 1) \\
& \quad \quad \quad * \mathbb{N}\mathbb{R} \ ((n + 1) \ !)^{-1})
\end{aligned}$$

**exp-series-convergence\_thm**

$$\begin{aligned}
& \vdash \forall \ x \\
& \bullet 0. < x \\
& \Rightarrow (\exists \ c \\
& \bullet (\lambda \ m \\
& \quad \bullet \text{PowerSeries} \ (\lambda \ m \bullet \text{Abs} \ (\mathbb{N}\mathbb{R} \ (m \ !)^{-1})) \ m \ x) \\
& \quad \rightarrow c)
\end{aligned}$$

**exp-consistency\_thm**

$$\begin{aligned}
& \vdash \exists \ f \\
& \bullet (\forall \ y \bullet (f \ Deriv \ f \ y) \ y) \\
& \quad \wedge f \ 0. = 1. \\
& \quad \wedge (\forall \ x \\
& \quad \bullet (\lambda \ m \bullet \text{PowerSeries} \ (\lambda \ m \bullet \mathbb{N}\mathbb{R} \ (m \ !)^{-1}) \ m \ x) \\
& \quad \quad \rightarrow f \ x)
\end{aligned}$$

**Exp-consistent**

$$\begin{aligned}
& \vdash \text{Consistent} \\
& \quad (\lambda \ \text{Exp}' \\
& \quad \bullet \text{Exp}' \ 0. = 1. \wedge (\forall \ x \bullet (\text{Exp}' \ Deriv \ \text{Exp}' \ x) \ x))
\end{aligned}$$

**exp\_def**

$$\vdash \text{Exp} \ 0. = 1. \wedge (\forall \ x \bullet (\text{Exp} \ Deriv \ \text{Exp} \ x) \ x)$$

**exp-unique\_thm**

$$\begin{aligned}
& \vdash \forall \ f \ g \ a \ b \ t \\
& \bullet (\forall \ x \bullet x \in \text{OpenInterval} \ a \ b \Rightarrow \neg f \ x = 0.) \\
& \quad \wedge (\forall \ x \bullet x \in \text{OpenInterval} \ a \ b \Rightarrow (f \ Deriv \ f \ x) \ x) \\
& \quad \wedge (\forall \ x \bullet x \in \text{OpenInterval} \ a \ b \Rightarrow (g \ Deriv \ g \ x) \ x) \\
& \quad \wedge t \in \text{OpenInterval} \ a \ b \\
& \Rightarrow (\forall \ x \\
& \quad \bullet x \in \text{OpenInterval} \ a \ b \\
& \quad \quad \Rightarrow g \ x = g \ t * f \ t^{-1} * f \ x)
\end{aligned}$$

**exp-cts\_thm**

$$\vdash \forall \ x \bullet \text{Exp} \ \text{Cts} \ x$$

**exp-¬\_eq\_0\_thm**

$$\vdash \forall \ x \bullet \neg \text{Exp} \ x = 0.$$

**exp-minus\_thm**

$$\vdash \forall \ x \bullet \text{Exp} \ (\sim x) = \text{Exp} \ x^{-1}$$

**exp\_0\_less\_thm**

$$\vdash \forall \ x \bullet 0. < \text{Exp} \ x$$

**exp-plus\_thm**

$$\vdash \forall \ x \ y \bullet \text{Exp} \ (x + y) = \text{Exp} \ x * \text{Exp} \ y$$

**exp-clauses**

$$\begin{aligned}
& \vdash \text{Exp} \ 0. = 1. \\
& \quad \wedge (\forall \ x \bullet \text{Exp} \ (\sim x) = \text{Exp} \ x^{-1})
\end{aligned}$$

$$\begin{aligned} & \wedge (\forall x y \bullet \text{Exp } (x + y) = \text{Exp } x * \text{Exp } y) \\ & \wedge (\forall x \bullet 0. < \text{Exp } x) \\ & \wedge (\forall x \bullet 0. \leq \text{Exp } x) \end{aligned}$$

**exp\_less\_mono\_thm**

$$\vdash \forall x y \bullet x < y \Leftrightarrow \text{Exp } x < \text{Exp } y$$

**exp\_power\_series\_thm**

$$\begin{aligned} & \vdash \forall x \\ & \bullet (\lambda m \bullet \text{PowerSeries } (\lambda m \bullet \text{NIR } (m !) ^{-1}) m x) \\ & \quad \rightarrow \text{Exp } x \end{aligned}$$

**Log\_consistent**

$$\vdash \text{Consistent } (\lambda \text{Log}' \bullet \forall x \bullet \text{Log}' (\text{Exp } x) = x)$$

**log\_def**

$$\vdash \forall x \bullet \text{Log } (\text{Exp } x) = x$$

**exp\_one\_one\_thm**

$$\vdash \text{OneOne } \text{Exp}$$

**exp\_ℝ.N\_exp\_thm**

$$\vdash \forall m x \bullet \text{Exp } (\text{NIR } m * x) = \text{Exp } x ^ m$$

**exp\_0\_less\_onto\_thm**

$$\vdash \forall x \bullet 0. < x \Rightarrow (\exists y \bullet x = \text{Exp } y)$$

**exp\_log\_thm**

$$\vdash \forall x \bullet 0. < x \Rightarrow \text{Exp } (\text{Log } x) = x$$

**log\_less\_mono\_thm**

$$\vdash \forall x y \bullet 0. < x \wedge x < y \Rightarrow \text{Log } x < \text{Log } y$$

**log\_cts\_thm**

$$\vdash \forall x \bullet 0. < x \Rightarrow \text{Log } \text{Cts } x$$

**log\_deriv\_thm**

$$\vdash \forall x \bullet 0. < x \Rightarrow (\text{Log } \text{Deriv } x^{-1}) x$$

**log\_clauses**

$$\begin{aligned} & \vdash \text{Log } 1. = 0. \\ & \wedge (\forall x \bullet 0. < x \Rightarrow \text{Log } (x^{-1}) = \sim (\text{Log } x)) \\ & \wedge (\forall x y \\ & \bullet 0. < x \wedge 0. < y \Rightarrow \text{Log } (x * y) = \text{Log } x + \text{Log } y) \end{aligned}$$

**ℝ.N\_exp\_exp\_log\_thm**

$$\vdash \forall m x \bullet 0. < x \Rightarrow x ^ m = \text{Exp } (\text{NIR } m * \text{Log } x)$$

**positive\_root\_thm**

$$\vdash \forall m x \bullet \neg m = 0 \wedge 0. < x \Rightarrow (\exists y \bullet 0. < y \wedge y ^ m = x)$$

**square\_root\_thm**

$$\vdash \forall x \bullet 0. < x \Leftrightarrow (\exists y \bullet 0. < y \wedge y ^ 2 = x)$$

**square\_root\_thm1**

$$\vdash \forall x \bullet 0. \leq x \Leftrightarrow (\exists y \bullet 0. \leq y \wedge y ^ 2 = x)$$

**square\_root\_unique\_thm**

$$\vdash \forall x y \bullet x ^ 2 = y ^ 2 \Leftrightarrow x = y \vee x = \sim y$$

**square\_square\_root\_mono\_thm**

$$\vdash \forall x y \bullet 0. < x \wedge 0. < y \Rightarrow (x ^ 2 < y ^ 2 \Leftrightarrow x < y)$$

**sin\_series\_convergence\_thm**

$$\begin{aligned} & \vdash \forall x \\ & \bullet 0. < x \\ & \quad \Rightarrow (\exists c \\ & \bullet (\lambda m \\ & \quad \bullet \text{PowerSeries} \\ & \quad \quad (\lambda m \\ & \quad \quad \bullet \text{Abs} \\ & \quad \quad \quad (\text{if } m \text{ Mod } 2 = 0 \\ & \quad \quad \quad \text{then } 0. \\ & \quad \quad \quad \text{else} \end{aligned}$$

$$\sim 1. \wedge (m \text{ Div } 2) \\ * \text{NR} (m !)^{-1})$$

$$\begin{array}{c} m \\ x) \\ \rightarrow c) \end{array}$$

**even\_odd.thm**  $\vdash \forall m \bullet \exists n \bullet m = 2 * n \vee m = 2 * n + 1$   
**mod\_2\_div\_2.thm**

$$\vdash \forall n \\ \bullet (2 * n) \text{ Mod } 2 = 0 \\ \wedge (2 * n + 1) \text{ Mod } 2 = 1 \\ \wedge (2 * n) \text{ Div } 2 = n \\ \wedge (2 * n + 1) \text{ Div } 2 = n$$

**mod\_2\_cases.thm**

$$\vdash \forall m \bullet m \text{ Mod } 2 = 0 \vee m \text{ Mod } 2 = 1$$

**mod\_2\_0.R.N.exp.thm**

$$\vdash \forall m \ x \bullet m \text{ Mod } 2 = 0 \Rightarrow \sim x \wedge m = x \wedge m$$

**mod\_2\_1.R.N.exp.thm**

$$\vdash \forall m \ x \bullet m \text{ Mod } 2 = 1 \Rightarrow \sim x \wedge m = \sim (x \wedge m)$$

**power\_series\_even.thm**

$$\vdash \forall s \ n \ x \\ \bullet (\forall m \bullet m \text{ Mod } 2 = 1 \Rightarrow s \ m = 0.) \\ \Rightarrow \text{PowerSeries } s \ n (\sim x) = \text{PowerSeries } s \ n \ x$$

**power\_series\_odd.thm**

$$\vdash \forall s \ n \ x \\ \bullet (\forall m \bullet m \text{ Mod } 2 = 0 \Rightarrow s \ m = 0.) \\ \Rightarrow \text{PowerSeries } s \ n (\sim x) = \sim (\text{PowerSeries } s \ n \ x)$$

**sin\_deriv\_coeffs.thm**

$$\vdash \forall m \\ \bullet \text{NR} (m + 1) \\ * \text{NR} (m + 2) \\ * (\text{if } (m + 2) \text{ Mod } 2 = 0 \\ \text{then } 0. \\ \text{else} \\ \sim 1. \wedge ((m + 2) \text{ Div } 2) \\ * \text{NR} ((m + 2) !)^{-1}) \\ = \sim \\ (\text{if } m \text{ Mod } 2 = 0 \\ \text{then } 0. \\ \text{else } \sim 1. \wedge (m \text{ Div } 2) * \text{NR} (m !)^{-1})$$

**sin\_cos\_consistency.thm**

$$\vdash \exists s \ c \\ \bullet (\forall y \bullet (s \text{ Deriv } c \ y) \ y) \\ \wedge (\forall y \bullet (c \text{ Deriv } \sim (s \ y)) \ y) \\ \wedge s \ 0. = 0. \\ \wedge c \ 0. = 1. \\ \wedge (\forall x \\ \bullet (\lambda n \\ \bullet \text{PowerSeries} \\ (\lambda m \\ \bullet \text{if } m \text{ Mod } 2 = 0 \\ \text{then } 0.$$

$$\begin{aligned}
& \text{else} \\
& \quad \sim 1. \wedge (m \text{ Div } 2) * \text{NIR } (m !) ^{-1}) \\
& \quad n \\
& \quad x) \\
& \quad \rightarrow s x) \\
& \wedge (\forall x \\
& \bullet (\lambda n \\
& \quad \bullet \text{PowerSeries} \\
& \quad (\lambda m \\
& \quad \quad \bullet \text{NIR } (m + 1) \\
& \quad \quad * (\text{if } (m + 1) \text{ Mod } 2 = 0 \\
& \quad \quad \text{then } 0. \\
& \quad \quad \text{else} \\
& \quad \quad \quad \sim 1. \wedge ((m + 1) \text{ Div } 2) \\
& \quad \quad \quad * \text{NIR } ((m + 1) !) ^{-1})) \\
& \quad n \\
& \quad x) \\
& \quad \rightarrow c x)
\end{aligned}$$

**Sin\_consistent**

**Cos\_consistent**

$$\begin{aligned}
& \vdash \text{Consistent} \\
& (\lambda (\text{Sin}', \text{Cos}') \\
& \quad \bullet \text{Sin}' 0. = 0. \\
& \quad \wedge \text{Cos}' 0. = 1. \\
& \quad \wedge (\forall x \bullet (\text{Sin}' \text{ Deriv } \text{Cos}' x) x) \\
& \quad \wedge (\forall x \bullet (\text{Cos}' \text{ Deriv } \sim (\text{Sin}' x)) x))
\end{aligned}$$

**sin\_def**

**cos\_def**

$$\begin{aligned}
& \vdash \text{Sin } 0. = 0. \\
& \quad \wedge \text{Cos } 0. = 1. \\
& \quad \wedge (\forall x \bullet (\text{Sin } \text{ Deriv } \text{Cos } x) x) \\
& \quad \wedge (\forall x \bullet (\text{Cos } \text{ Deriv } \sim (\text{Sin } x)) x)
\end{aligned}$$

**sin\_cos\_cts\_thm**

$$\vdash \forall x \bullet \text{Sin } \text{Cts } x \wedge \text{Cos } \text{Cts } x$$

**cos\_squared\_plus\_sin\_squared\_thm**

$$\vdash \forall x \bullet \text{Cos } x \wedge 2 + \text{Sin } x \wedge 2 = 1.$$

**sin\_cos\_unique\_lemma1**

$$\begin{aligned}
& \vdash \forall f g x \\
& \quad \bullet (\forall x \bullet (f \text{ Deriv } g x) x) \wedge (\forall x \bullet (g \text{ Deriv } \sim (f x)) x) \\
& \quad \Rightarrow \text{Cos } x * g x + \text{Sin } x * f x = g 0. \\
& \quad \wedge \text{Cos } x * f x - \text{Sin } x * g x = f 0.
\end{aligned}$$

**sin\_cos\_unique\_lemma2**

$$\begin{aligned}
& \vdash \forall f g x \\
& \quad \bullet (\forall x \bullet (f \text{ Deriv } g x) x) \wedge (\forall x \bullet (g \text{ Deriv } \sim (f x)) x) \\
& \quad \Rightarrow f x = f 0. * \text{Cos } x + g 0. * \text{Sin } x \\
& \quad \wedge g x = g 0. * \text{Cos } x - f 0. * \text{Sin } x
\end{aligned}$$

**sin\_cos\_unique\_thm**

$$\begin{aligned}
& \vdash \forall f g x \\
& \quad \bullet f 0. = 0. \\
& \quad \wedge g 0. = 1. \\
& \quad \wedge (\forall x \bullet (f \text{ Deriv } g x) x) \\
& \quad \wedge (\forall x \bullet (g \text{ Deriv } \sim (f x)) x)
\end{aligned}$$

$$\Rightarrow f x = \text{Sin } x \wedge g x = \text{Cos } x$$

**sin\_cos\_power\_series\_thm**

$$\begin{aligned} &\vdash \forall x \\ &\bullet (\lambda n \\ &\quad \bullet \text{PowerSeries} \\ &\quad (\lambda m \\ &\quad \quad \bullet \text{if } m \text{ Mod } 2 = 0 \\ &\quad \quad \text{then } 0. \\ &\quad \quad \text{else } \sim 1. \wedge (m \text{ Div } 2) * \text{NIR } (m !) ^{-1}) \\ &\quad n \\ &\quad x) \\ &\rightarrow \text{Sin } x \\ &\wedge (\lambda n \\ &\quad \bullet \text{PowerSeries} \\ &\quad (\lambda m \\ &\quad \quad \bullet \text{NIR } (m + 1) \\ &\quad \quad * (\text{if } (m + 1) \text{ Mod } 2 = 0 \\ &\quad \quad \text{then } 0. \\ &\quad \quad \text{else} \\ &\quad \quad \quad \sim 1. \wedge ((m + 1) \text{ Div } 2) \\ &\quad \quad \quad * \text{NIR } ((m + 1) !) ^{-1})) \\ &\quad n \\ &\quad x) \\ &\rightarrow \text{Cos } x \end{aligned}$$

**sin\_cos\_plus\_thm**

$$\begin{aligned} &\vdash \forall x y \\ &\bullet \text{Sin } (x + y) = \text{Sin } x * \text{Cos } y + \text{Cos } x * \text{Sin } y \\ &\quad \wedge \text{Cos } (x + y) = \text{Cos } x * \text{Cos } y - \text{Sin } x * \text{Sin } y \end{aligned}$$

**sin\_cos\_minus\_thm**

$$\vdash \forall x \bullet \text{Sin } (\sim x) = \sim (\text{Sin } x) \wedge \text{Cos } (\sim x) = \text{Cos } x$$

**sin\_0\_group\_thm**

$$\begin{aligned} &\vdash 0. \in \{x | \text{Sin } x = 0.\} \\ &\quad \wedge (\forall x y \\ &\quad \quad \bullet x \in \{x | \text{Sin } x = 0.\} \wedge y \in \{x | \text{Sin } x = 0.\} \\ &\quad \quad \Rightarrow x + y \in \{x | \text{Sin } x = 0.\}) \\ &\quad \wedge (\forall x \bullet x \in \{x | \text{Sin } x = 0.\} \Rightarrow \sim x \in \{x | \text{Sin } x = 0.\}) \end{aligned}$$

**sin\_0\_NIR\_times\_thm**

$$\vdash \forall x m \bullet \text{Sin } x = 0. \Rightarrow \text{Sin } (\text{NIR } m * x) = 0.$$

**sin\_eq\_cos\_sin\_cos\_twice\_thm**

$$\begin{aligned} &\vdash \forall x \\ &\quad \bullet \text{Sin } x = \text{Cos } x \\ &\quad \Rightarrow \text{Sin } (2. * x) = 1. \wedge \text{Cos } (2. * x) = 0. \end{aligned}$$

**sum\_squares\_abs\_bound\_thm**

$$\vdash \forall x y z \bullet x^2 + y^2 = z^2 \Rightarrow \text{Abs } x \leq \text{Abs } z$$

**sum\_squares\_abs\_bound\_thm1**

$$\vdash \forall x y \bullet x^2 + y^2 = 1. \Rightarrow \text{Abs } x \leq 1.$$

**abs\_sin\_abs\_cos\_le\_1\_thm**

$$\vdash \forall x \bullet \text{Abs } (\text{Sin } x) \leq 1. \wedge \text{Abs } (\text{Cos } x) \leq 1.$$

**cos\_positive\_estimate\_thm**

$$\vdash \forall x \bullet x \in \text{OpenInterval } 0. 1. \Rightarrow 0. < \text{Cos } x$$

**sin\_positive\_estimate\_thm**

$\vdash \forall x \bullet x \in \text{OpenInterval } 0. 2. \Rightarrow 0. < \text{Sin } x$   
**cos\_greater\_root\_2\_thm**  
 $\vdash \forall t$   
 $\bullet 0. < t \wedge (\forall x \bullet 0. < x \wedge x < t \Rightarrow 1 / 2 < \text{Cos } x \wedge 2) \Rightarrow t \wedge 2 \leq 2.$

**cos\_squared\_eq\_half\_thm**  
 $\vdash \forall e$   
 $\bullet 0. < e$   
 $\Rightarrow (\exists x$   
 $\bullet 0. < x$   
 $\wedge x \wedge 2 < 2. + e$   
 $\wedge 0. < \text{Cos } x$   
 $\wedge \text{Cos } x \wedge 2 = 1 / 2)$

**sin\_eq\_cos\_exists\_thm**  
 $\vdash \exists x \bullet x \in \text{OpenInterval } 0. 2. \wedge \text{Sin } x = \text{Cos } x$

**sin\_positive\_zero\_thm**  
 $\vdash \exists x \bullet 0. < x \wedge \text{Sin } x = 0.$

**$\mathbb{R}$ \_discrete\_subgroup\_thm**  
 $\vdash \forall G h a$   
 $\bullet 0. \in G$   
 $\wedge (\forall g h \bullet g \in G \wedge h \in G \Rightarrow g + h \in G)$   
 $\wedge (\forall g \bullet g \in G \Rightarrow \sim g \in G)$   
 $\wedge 0. < h$   
 $\wedge h \in G$   
 $\wedge 0. < a$   
 $\wedge (\forall h \bullet 0. < h \wedge h \in G \Rightarrow a < h)$   
 $\Rightarrow (\exists g$   
 $\bullet 0. < g$   
 $\wedge g \in G$   
 $\wedge (\forall h$   
 $\bullet 0. < h \wedge h \in G \Rightarrow (\exists m \bullet h = \mathbb{N}R m * g)))$

**pi\_consistency\_thm**  
 $\vdash \exists pi$   
 $\bullet 0. < pi$   
 $\wedge \text{Sin } pi = 0.$   
 $\wedge (\forall x$   
 $\bullet 0. < x \wedge \text{Sin } x = 0. \Rightarrow (\exists m \bullet x = \mathbb{N}R m * pi))$

**ArchimedesConstant\_consistent**  
 $\vdash \text{Consistent}$   
 $(\lambda \text{ArchimedesConstant}'$   
 $\bullet 0. < \text{ArchimedesConstant}'$   
 $\wedge (\forall x$   
 $\bullet \text{Sin } x = 0.$   
 $\Leftrightarrow (\exists m \bullet x = \mathbb{N}R m * \text{ArchimedesConstant}')$   
 $\vee (\exists m$   
 $\bullet x = \sim (\mathbb{N}R m * \text{ArchimedesConstant}'))))$

**$\pi$ \_def**  
 $\vdash 0. < \pi$   
 $\wedge (\forall x$   
 $\bullet \text{Sin } x = 0.$   
 $\Leftrightarrow (\exists m \bullet x = \mathbb{N}R m * \pi)$   
 $\vee (\exists m \bullet x = \sim (\mathbb{N}R m * \pi)))$



**NR\_plus\_1\_times\_bound\_thm**

$$\begin{aligned} &\vdash (\forall c m \bullet 0. < c \Rightarrow c \leq \text{NR } (m + 1) * c) \\ &\quad \wedge (\forall c m \bullet c < 0. \Rightarrow \text{NR } (m + 1) * c \leq c) \end{aligned}$$

**NR\_0\_less\_not\_NR\_times\_pi\_thm**

$$\vdash \forall x m \bullet 0. < x \Rightarrow \neg x = \sim (\text{NR } m * \pi)$$

**sin\_not\_eq\_0\_thm**

$$\vdash \forall x \bullet 0. < x \wedge x < \pi \Rightarrow \neg \text{Sin } x = 0.$$

**sin\_0\_pi\_0\_less\_thm**

$$\vdash \forall x \bullet x \in \text{OpenInterval } 0. \pi \Rightarrow 0. < \text{Sin } x$$

**sin\_cos\_pi\_over\_2\_thm**

$$\vdash \text{Sin } (1 / 2 * \pi) = 1. \wedge \text{Cos } (1 / 2 * \pi) = 0.$$

**sin\_cos\_plus\_pi\_over\_2\_thm**

$$\begin{aligned} &\vdash \forall x \\ &\bullet \text{Sin } (x + 1 / 2 * \pi) = \text{Cos } x \\ &\quad \wedge \text{Cos } (x + 1 / 2 * \pi) = \sim (\text{Sin } x) \end{aligned}$$

**cos\_not\_eq\_0\_thm**

$$\vdash \forall x \bullet \sim (1 / 2) * \pi < x \wedge x < 1 / 2 * \pi \Rightarrow \neg \text{Cos } x = 0.$$

**cos\_eq\_0\_thm**

$$\begin{aligned} &\vdash \forall x \\ &\bullet \text{Cos } x = 0. \\ &\quad \Leftrightarrow (\exists m \bullet x = 1 / 2 * \pi + \text{NR } m * \pi) \\ &\quad \vee (\exists m \bullet x = 1 / 2 * \pi - \text{NR } m * \pi) \end{aligned}$$

**sin\_cos\_plus\_pi\_thm**

$$\begin{aligned} &\vdash \forall x \\ &\bullet \text{Sin } (x + \pi) = \sim (\text{Sin } x) \wedge \text{Cos } (x + \pi) = \sim (\text{Cos } x) \end{aligned}$$

**sin\_cos\_pi\_thm**

$$\vdash \text{Sin } \pi = 0. \wedge \text{Cos } \pi = \sim 1.$$

**sin\_cos\_plus\_2\_pi\_thm**

$$\begin{aligned} &\vdash \forall x \\ &\bullet \text{Sin } (x + 2. * \pi) = \text{Sin } x \wedge \text{Cos } (x + 2. * \pi) = \text{Cos } x \end{aligned}$$

**sin\_cos\_2\_pi\_thm**

$$\vdash \text{Sin } (2. * \pi) = 0. \wedge \text{Cos } (2. * \pi) = 1.$$

**sin\_cos\_plus\_even\_times\_pi\_thm**

$$\begin{aligned} &\vdash \forall x m \\ &\bullet \text{Sin } (x + \text{NR } (2 * m) * \pi) = \text{Sin } x \\ &\quad \wedge \text{Cos } (x + \text{NR } (2 * m) * \pi) = \text{Cos } x \end{aligned}$$

**sin\_cos\_even\_times\_pi\_thm**

$$\begin{aligned} &\vdash \forall m \\ &\bullet \text{Sin } (\text{NR } (2 * m) * \pi) = 0. \\ &\quad \wedge \text{Cos } (\text{NR } (2 * m) * \pi) = 1. \end{aligned}$$

**sin\_cos\_period\_thm**

$$\begin{aligned} &\vdash \forall x y \\ &\bullet x < y \\ &\quad \Rightarrow (\text{Sin } x = \text{Sin } y \wedge \text{Cos } x = \text{Cos } y) \\ &\quad \Leftrightarrow (\exists m \bullet y = x + \text{NR } (2 * m) * \pi) \end{aligned}$$

**sin\_cos\_onto\_unit\_circle\_thm**

$$\begin{aligned} &\vdash \forall x y \\ &\bullet x^2 + y^2 = 1. \\ &\quad \Rightarrow (\exists z \\ &\quad \bullet 0. \leq z \wedge z < 2. * \pi \wedge x = \text{Cos } z \wedge y = \text{Sin } z) \end{aligned}$$

**sin\_cos\_onto\_unit\_circle\_thm1**

$$\vdash \forall x y$$

- $x^2 + y^2 = 1.$   
 $\Rightarrow (\exists_1 z$   
 •  $0. \leq z \wedge z < 2. * \pi \wedge x = \text{Cos } z \wedge y = \text{Sin } z)$

**lim\_right\_lim\_seq\_thm**

- $\vdash \forall f c x$
- $(f \text{ +\#-} \rightarrow c) x$   
 $\Leftrightarrow (\forall s$   
 •  $s \rightarrow x \wedge (\forall m. x < s m) \Rightarrow (\lambda m. f (s m)) \rightarrow c)$

**lim\_fun\_lim\_right\_thm**

- $\vdash \forall f c x$
- $(f \text{ --} \rightarrow c) x$   
 $\Leftrightarrow (f \text{ +\#-} \rightarrow c) x \wedge ((\lambda y. f (\sim y)) \text{ +\#-} \rightarrow c) (\sim x)$

**const\_lim\_right\_thm**

- $\vdash \forall c x. ((\lambda x. c) \text{ +\#-} \rightarrow c) x$

**id\_lim\_right\_thm**

- $\vdash \forall x. ((\lambda x. x) \text{ +\#-} \rightarrow x) x$

**plus\_lim\_right\_thm**

- $\vdash \forall f1 c1 f2 c2 x$
- $(f1 \text{ +\#-} \rightarrow c1) x \wedge (f2 \text{ +\#-} \rightarrow c2) x$   
 $\Rightarrow ((\lambda x. f1 x + f2 x) \text{ +\#-} \rightarrow c1 + c2) x$

**times\_lim\_right\_thm**

- $\vdash \forall f1 c1 f2 c2 x$
- $(f1 \text{ +\#-} \rightarrow c1) x \wedge (f2 \text{ +\#-} \rightarrow c2) x$   
 $\Rightarrow ((\lambda x. f1 x * f2 x) \text{ +\#-} \rightarrow c1 * c2) x$

**poly\_lim\_right\_thm**

- $\vdash \forall f g c x$
- $f \in \text{PolyFunc} \wedge (g \text{ +\#-} \rightarrow c) x$   
 $\Rightarrow ((\lambda x. f (g x)) \text{ +\#-} \rightarrow f c) x$

**recip\_lim\_right\_thm**

- $\vdash \forall f c x$
- $(f \text{ +\#-} \rightarrow c) x \wedge \neg c = 0.$   
 $\Rightarrow ((\lambda x. f x^{-1}) \text{ +\#-} \rightarrow c^{-1}) x$

**lim\_right\_unique\_thm**

- $\vdash \forall f c d x. (f \text{ +\#-} \rightarrow c) x \wedge (f \text{ +\#-} \rightarrow d) x \Rightarrow c = d$

**cts\_lim\_right\_thm**

- $\vdash \forall f c x. f \text{ Cts } x \wedge (f \text{ +\#-} \rightarrow c) x \Rightarrow f x = c$

**lim\_infinity\_lim\_seq\_thm**

- $\vdash \forall f c$
- $f \text{ --+\#} > c$   
 $\Leftrightarrow (\forall s. (\forall m. \text{NR } m \leq s m) \Rightarrow (\lambda m. f (s m)) \rightarrow c)$

**lim\_infinity\_lim\_right\_thm**

- $\vdash \forall f c. f \text{ --+\#} > c \Leftrightarrow ((\lambda x. f (x^{-1})) \text{ +\#-} \rightarrow c) 0.$

**lim\_right\_lim\_infinity\_thm**

- $\vdash \forall f c. ((\lambda x. f (x^{-1})) \text{ +\#-} \rightarrow c) 0. \Leftrightarrow f \text{ --+\#} > c$

**const\_lim\_infinity\_thm**

- $\vdash \forall c. (\lambda x. c) \text{ --+\#} > c$

**id\_lim\_infinity\_thm**

- $\vdash \forall c. \neg (\lambda x. x) \text{ --+\#} > c$

**plus\_lim\_infinity\_thm**

- $\vdash \forall f1 c1 f2 c2$
- $f1 \text{ --+\#} > c1 \wedge f2 \text{ --+\#} > c2$

$\Rightarrow (\lambda x \bullet f1\ x + f2\ x) \text{ --}+\#\text{>} c1 + c2$

**times\_lim\_infinity\_thm**  
 $\vdash \forall f1\ c1\ f2\ c2$   
 $\bullet f1 \text{ --}+\#\text{>} c1 \wedge f2 \text{ --}+\#\text{>} c2$   
 $\Rightarrow (\lambda x \bullet f1\ x * f2\ x) \text{ --}+\#\text{>} c1 * c2$

**poly\_lim\_infinity\_thm**  
 $\vdash \forall f\ g\ c$   
 $\bullet f \in PolyFunc \wedge g \text{ --}+\#\text{>} c \Rightarrow (\lambda x \bullet f\ (g\ x)) \text{ --}+\#\text{>} f\ c$

**recip\_lim\_infinity\_thm**  
 $\vdash \forall f\ c$   
 $\bullet f \text{ --}+\#\text{>} c \wedge \neg c = 0. \Rightarrow (\lambda x \bullet f\ x^{-1}) \text{ --}+\#\text{>} c^{-1}$

**recip\_lim\_infinity\_0\_thm**  
 $\vdash (\lambda x \bullet x^{-1}) \text{ --}+\#\text{>} 0.$

**lim\_infinity\_unique\_thm**  
 $\vdash \forall f\ c\ d \bullet f \text{ --}+\#\text{>} c \wedge f \text{ --}+\#\text{>} d \Rightarrow c = d$

**div\_infinity\_pos\_thm**  
 $\vdash \forall f$   
 $\bullet f \text{ --}+\#\text{>}+\# \Rightarrow (\exists b \bullet 0. < b \wedge (\forall x \bullet b < x \Rightarrow 0. < f\ x))$

**const\_plus\_div\_infinity\_thm**  
 $\vdash \forall f\ c \bullet f \text{ --}+\#\text{>}+\# \Rightarrow (\lambda x \bullet c + f\ x) \text{ --}+\#\text{>}+\#$

**id\_div\_infinity\_thm**  
 $\vdash (\lambda x \bullet x) \text{ --}+\#\text{>}+\#$

**plus\_div\_infinity\_thm**  
 $\vdash \forall f\ g \bullet f \text{ --}+\#\text{>}+\# \wedge g \text{ --}+\#\text{>}+\# \Rightarrow (\lambda x \bullet f\ x + g\ x) \text{ --}+\#\text{>}+\#$

**const\_times\_div\_infinity\_thm**  
 $\vdash \forall f\ c \bullet f \text{ --}+\#\text{>}+\# \wedge 0. < c \Rightarrow (\lambda x \bullet c * f\ x) \text{ --}+\#\text{>}+\#$

**times\_div\_infinity\_thm**  
 $\vdash \forall f\ g \bullet f \text{ --}+\#\text{>}+\# \wedge g \text{ --}+\#\text{>}+\# \Rightarrow (\lambda x \bullet f\ x * g\ x) \text{ --}+\#\text{>}+\#$

**power\_div\_infinity\_thm**  
 $\vdash \forall m \bullet (\lambda x \bullet x^{(m+1)}) \text{ --}+\#\text{>}+\#$

**less\_div\_infinity\_thm**  
 $\vdash \forall f\ g\ a$   
 $\bullet f \text{ --}+\#\text{>}+\# \wedge (\forall x \bullet a < x \Rightarrow f\ x < g\ x) \Rightarrow g \text{ --}+\#\text{>}+\#$

**bounded\_deriv\_div\_infinity\_thm**  
 $\vdash \forall f\ df\ a\ c$   
 $\bullet 0. < c$   
 $\wedge (\forall x \bullet a \leq x \Rightarrow (f\ Deriv\ df\ x)\ x)$   
 $\wedge (\forall x \bullet a \leq x \Rightarrow c < df\ x)$   
 $\Rightarrow f \text{ --}+\#\text{>}+\#$

**exp\_div\_infinity\_thm**  
 $\vdash Exp \text{ --}+\#\text{>}+\#$

**log\_div\_infinity\_thm**  
 $\vdash Log \text{ --}+\#\text{>}+\#$

**div\_infinity\_times\_recip\_thm**  
 $\vdash \forall f\ a\ e$   
 $\bullet f \text{ --}+\#\text{>}+\# \wedge 0. < a \wedge 0. < e$   
 $\Rightarrow (\exists b \bullet \forall x \bullet b < x \Rightarrow a * f\ x^{-1} < e)$

**div\_infinity\_lim\_right\_thm**  
 $\vdash \forall f$   
 $\bullet f \text{ --}+\#\text{>}+\#$   
 $\Leftrightarrow ((\lambda x \bullet f\ (x^{-1})^{-1}) \text{ --}+\#\text{>} 0.) 0.$

$\wedge (\exists b \bullet \forall x \bullet b < x \Rightarrow 0. < f x)$

**div\_infinity\_lim\_infinity\_thm**  
 $\vdash \forall f$   
 $\bullet f \text{ --}\#>\text{+}\#$   
 $\Leftrightarrow (\lambda x \bullet f x^{-1}) \text{ --}\#> 0.$   
 $\wedge (\exists b \bullet \forall x \bullet b < x \Rightarrow 0. < f x)$

**comp\_div\_infinity\_thm**  
 $\vdash \forall f g \bullet f \text{ --}\#>\text{+}\# \wedge g \text{ --}\#>\text{+}\# \Rightarrow (\lambda x \bullet f (g x)) \text{ --}\#>\text{+}\#$

**rolle\_thm1**  
 $\vdash \forall f df a b$   
 $\bullet a < b$   
 $\wedge (\forall x \bullet a \leq x \wedge x < b \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } df x) x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow \neg df x = 0.)$   
 $\Rightarrow (\forall x \bullet a < x \wedge x < b \Rightarrow \neg f x = f a)$

**l'hopital\_lim\_right\_thm**  
 $\vdash \forall f df g dg a b c$   
 $\bullet a < b$   
 $\wedge f a = 0.$   
 $\wedge g a = 0.$   
 $\wedge (\forall x \bullet a \leq x \wedge x < b \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (f \text{ Deriv } df x) x)$   
 $\wedge (\forall x \bullet a \leq x \wedge x < b \Rightarrow g \text{ Cts } x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (g \text{ Deriv } dg x) x)$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow \neg dg x = 0.)$   
 $\wedge ((\lambda x \bullet df x * dg x^{-1}) \text{ +}\#\text{--}\> c) a$   
 $\Rightarrow ((\lambda x \bullet f x * g x^{-1}) \text{ +}\#\text{--}\> c) a$

**cts\_comp\_minus\_thm**  
 $\vdash \forall f x \bullet (\lambda x \bullet f (\sim x)) \text{ Cts } x \Leftrightarrow f \text{ Cts } \sim x$

**deriv\_comp\_minus\_thm**  
 $\vdash \forall f df x$   
 $\bullet ((\lambda x \bullet f (\sim x)) \text{ Deriv } \sim (df (\sim x))) x$   
 $\Leftrightarrow (f \text{ Deriv } df (\sim x)) (\sim x)$

**l'hopital\_lim\_fun\_thm**  
 $\vdash \forall f df g dg a b c x$   
 $\bullet a < x$   
 $\wedge x < b$   
 $\wedge f x = 0.$   
 $\wedge g x = 0.$   
 $\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow f \text{ Cts } y)$   
 $\wedge (\forall y$   
 $\bullet a < y \wedge y < b \wedge \neg y = x \Rightarrow (f \text{ Deriv } df y) y)$   
 $\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow g \text{ Cts } y)$   
 $\wedge (\forall y$   
 $\bullet a < y \wedge y < b \wedge \neg y = x \Rightarrow (g \text{ Deriv } dg y) y)$   
 $\wedge (\forall y \bullet a < y \wedge y < b \Rightarrow \neg dg y = 0.)$   
 $\wedge ((\lambda y \bullet df y * dg y^{-1}) \text{ --}\#\text{--}\> c) x$   
 $\Rightarrow ((\lambda y \bullet f y * g y^{-1}) \text{ --}\#\text{--}\> c) x$

**l'hopital\_lim\_infinity\_thm**  
 $\vdash \forall f df g dg a c$   
 $\bullet g \text{ --}\#>\text{+}\#$   
 $\wedge (\forall x \bullet a < x \Rightarrow (f \text{ Deriv } df x) x)$

$$\begin{aligned}
& \wedge (\forall x \bullet a < x \Rightarrow (g \text{ Deriv } dg \ x) \ x) \\
& \wedge (\forall x \bullet a < x \Rightarrow \neg dg \ x = 0.) \\
& \wedge (\lambda x \bullet df \ x * dg \ x^{-1}) \text{ --+\#> } c \\
& \Rightarrow (\lambda x \bullet f \ x * g \ x^{-1}) \text{ --+\#> } c
\end{aligned}$$

**cts\_deriv\_deriv\_thm**

$$\begin{aligned}
& \vdash \forall f \ df \ a \ b \ t \\
& \bullet (\forall x \\
& \quad \bullet x \in \text{OpenInterval } a \ b \ \setminus \ \{t\} \\
& \quad \Rightarrow (f \ \text{Deriv } df \ x) \ x) \\
& \wedge (\forall x \bullet x \in \text{OpenInterval } a \ b \Rightarrow df \ \text{Cts } x) \\
& \wedge f \ \text{Cts } t \\
& \wedge t \in \text{OpenInterval } a \ b \\
& \Rightarrow (\forall x \bullet x \in \text{OpenInterval } a \ b \Rightarrow (f \ \text{Deriv } df \ x) \ x)
\end{aligned}$$

**lim\_fun\_id\_over\_sin\_thm**

$$\vdash ((\lambda x \bullet x * \text{Sin } x^{-1}) \text{ --> } 1.) \ 0.$$

**lim\_infinity\_recip\_exp\_thm**

$$\vdash (\lambda x \bullet \text{Exp } x^{-1}) \text{ --+\#> } 0.$$

**lim\_infinity\_id\_over\_exp\_thm**

$$\vdash (\lambda x \bullet x * \text{Exp } x^{-1}) \text{ --+\#> } 0.$$

**lim\_fun\_power\_over\_exp\_thm**

$$\vdash \forall m \bullet (\lambda x \bullet x \wedge m * \text{Exp } x^{-1}) \text{ --+\#> } 0.$$

**lim\_infinity\_log\_over\_id\_thm**

$$\vdash (\lambda x \bullet \text{Log } x * x^{-1}) \text{ --+\#> } 0.$$

**lim\_fun\_log\_over\_id\_minus\_1\_thm**

$$\vdash ((\lambda x \bullet \text{Log } x * (x - 1.)^{-1}) \text{ --> } 1.) \ 1.$$

**lim\_fun\_log\_1\_plus\_over\_id\_thm**

$$\vdash ((\lambda x \bullet \text{Log } (1. + x) * x^{-1}) \text{ --> } 1.) \ 0.$$

**lim\_fun\_e\_thm**

$$\vdash ((\lambda x \bullet \text{Exp } (x^{-1} * \text{Log } (1. + x))) \text{ --> } \text{Exp } 1.) \ 0.$$

**lim\_seq\_e\_thm**

$$\vdash (\lambda m \bullet (1. + \text{NR } m^{-1}) \wedge m) \text{ --> } \text{Exp } 1.$$

**$\mathbb{R}$ \_less\_mono\_↔\_thm**

$$\begin{aligned}
& \vdash \forall f \\
& \bullet (\forall x \ y \bullet x < y \Rightarrow f \ x < f \ y) \\
& \Leftrightarrow (\forall x \ y \bullet f \ x < f \ y \Leftrightarrow x < y)
\end{aligned}$$

**$\mathbb{R}$ \_less\_mono\_↔\_≤\_thm**

$$\begin{aligned}
& \vdash \forall f \\
& \bullet (\forall x \ y \bullet x < y \Rightarrow f \ x < f \ y) \\
& \Leftrightarrow (\forall x \ y \bullet f \ x \leq f \ y \Leftrightarrow x \leq y)
\end{aligned}$$

**$\mathbb{R}$ \_less\_mono\_one\_one\_thm**

$$\begin{aligned}
& \vdash \forall f \\
& \bullet (\forall x \ y \bullet x < y \Rightarrow f \ x < f \ y) \\
& \Rightarrow (\forall x \ y \bullet f \ x = f \ y \Leftrightarrow x = y)
\end{aligned}$$

**total\_inverse\_cts\_thm**

$$\begin{aligned}
& \vdash \forall f \ g \ x \\
& \bullet (\forall x \ y \bullet x < y \Rightarrow f \ x < f \ y) \\
& \quad \wedge (\forall x \bullet g \ (f \ x) = x) \\
& \quad \wedge (\forall y \bullet f \ (g \ y) = y) \\
& \Rightarrow g \ \text{Cts } f \ x
\end{aligned}$$

**total\_inverse\_thm**

$$\vdash \forall f$$

- $(\forall x y \bullet x < y \Rightarrow f x < f y) \wedge (\forall y \bullet \exists x \bullet f x = y)$   
 $\Rightarrow (\exists g$ 
  - $(\forall x \bullet g (f x) = x)$   
 $\wedge (\forall y \bullet f (g y) = y)$   
 $\wedge (\forall y \bullet g \text{ Cts } y)$   
 $\wedge (\forall x y \bullet x < y \Rightarrow g x < g y))$

**closed\_half\_infinite\_inverse\_thm1**

- $\vdash \forall f a$ 
  - $(\forall x \bullet a \leq x \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x y \bullet a \leq x \wedge x < y \Rightarrow f x < f y)$   
 $\wedge (\forall y \bullet f a \leq y \Rightarrow (\exists x \bullet a \leq x \wedge f x = y))$   
 $\Rightarrow (\exists g$ 
    - $(\forall x \bullet a \leq x \Rightarrow g (f x) = x)$   
 $\wedge (\forall y \bullet f a \leq y \Rightarrow f (g y) = y)$   
 $\wedge (\forall y \bullet g \text{ Cts } y)$   
 $\wedge (\forall x y \bullet x < y \Rightarrow g x < g y))$

**closed\_half\_infinite\_inverse\_thm**

- $\vdash \forall f df a$ 
  - $(\forall x \bullet a \leq x \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x \bullet a < x \Rightarrow (f \text{ Deriv } df x) x)$   
 $\wedge (\forall x \bullet a < x \Rightarrow 0. < df x)$   
 $\wedge (\forall y \bullet f a \leq y \Rightarrow (\exists x \bullet a \leq x \wedge f x = y))$   
 $\Rightarrow (\exists g$ 
    - $(\forall x \bullet a \leq x \Rightarrow g (f x) = x)$   
 $\wedge (\forall y \bullet f a \leq y \Rightarrow f (g y) = y)$   
 $\wedge (\forall y \bullet g \text{ Cts } y)$   
 $\wedge (\forall x y \bullet x < y \Rightarrow g x < g y))$

**cond\_cts\_thm**  $\vdash \forall f g y$

- $f \text{ Cts } y \wedge g \text{ Cts } y \wedge f y = g y$   
 $\Rightarrow (\lambda z \bullet \text{if } z \leq y \text{ then } f z \text{ else } g z) \text{ Cts } y$

**cond\_cts\_thm1**

- $\vdash \forall f g a b y x$ 
  - $(\forall x \bullet x \in \text{ClosedInterval } a y \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x \bullet x \in \text{ClosedInterval } y b \Rightarrow g \text{ Cts } x)$   
 $\wedge f y = g y$   
 $\wedge x \in \text{ClosedInterval } a b$   
 $\Rightarrow (\lambda z \bullet \text{if } z \leq y \text{ then } f z \text{ else } g z) \text{ Cts } x$

**closed\_interval\_inverse\_thm**

- $\vdash \forall f df a b$ 
  - $a < b$   
 $\wedge (\forall x \bullet x \in \text{ClosedInterval } a b \Rightarrow f \text{ Cts } x)$   
 $\wedge (\forall x$ 
    - $x \in \text{OpenInterval } a b \Rightarrow (f \text{ Deriv } df x) x)$   
 $\wedge (\forall x \bullet x \in \text{OpenInterval } a b \Rightarrow 0. < df x)$ $\Rightarrow (\exists g$ 
  - $(\forall x \bullet x \in \text{ClosedInterval } a b \Rightarrow g (f x) = x)$   
 $\wedge (\forall y$ 
    - $y \in \text{ClosedInterval } (f a) (f b)$   
 $\Rightarrow f (g y) = y)$   
 $\wedge (\forall y \bullet g \text{ Cts } y)$   
 $\wedge (\forall x y \bullet x < y \Rightarrow g x < g y))$

### *Sqrt\_consistent*

$\vdash$  *Consistent*  
 $(\lambda$  *Sqrt'*  
•  $\forall x$   
•  $0. \leq x$   
 $\Rightarrow 0. \leq$  *Sqrt'*  $x$   
 $\wedge$  *Sqrt'*  $x^2 = x$   
 $\wedge$  *Sqrt'* *Cts*  $x)$

*sqrt\_def*  $\vdash \forall x$   
•  $0. \leq x \Rightarrow 0. \leq$  *Sqrt*  $x \wedge$  *Sqrt*  $x^2 = x \wedge$  *Sqrt* *Cts*  $x$

*sqrt\_thm*  $\vdash \forall x \bullet 0. \leq x \Rightarrow$  *Sqrt*  $x^2 = x$

*sqrt\_0\_≤\_thm*  $\vdash \forall x \bullet 0. \leq x \Rightarrow 0. \leq$  *Sqrt*  $x$

*sqrt\_cts\_thm*  $\vdash \forall x \bullet 0. \leq x \Rightarrow$  *Sqrt* *Cts*  $x$

*sqrt\_square\_thm*  
 $\vdash \forall x \bullet$  *Sqrt*  $(x^2) =$  *Abs*  $x$

*square\_sqrt\_thm*  
 $\vdash \forall x y \bullet x = y^2 \Rightarrow$  *Sqrt*  $x =$  *Abs*  $y$

*sqrt\_0\_1\_thm*  $\vdash$  *Sqrt*  $0. = 0. \wedge$  *Sqrt*  $1. = 1.$

*square\_times\_thm*  
 $\vdash \forall x y \bullet (x * y)^2 = x^2 * y^2$

*sqrt\_times\_thm*  
 $\vdash \forall x y$   
•  $0. \leq x \wedge 0. \leq y \Rightarrow$  *Sqrt*  $(x * y) =$  *Sqrt*  $x *$  *Sqrt*  $y$

*square\_recip\_thm*  
 $\vdash \forall x \bullet \neg x = 0. \Rightarrow (x^{-1})^2 = (x^2)^{-1}$

*sqrt\_pos\_thm*  $\vdash \forall x \bullet 0. < x \Rightarrow 0. <$  *Sqrt*  $x$

*sqrt\_recip\_thm*  
 $\vdash \forall x \bullet 0. < x \Rightarrow$  *Sqrt*  $(x^{-1}) =$  *Sqrt*  $x^{-1}$

*sqrt\_exp\_log\_thm*  
 $\vdash \forall x \bullet 0. < x \Rightarrow$  *Sqrt*  $x =$  *Exp*  $(1 / 2 *$  *Log*  $x)$

*sqrt\_deriv\_thm*  
 $\vdash \forall x \bullet 0. < x \Rightarrow ($  *Sqrt* *Deriv*  $1 / 2 *$  *Sqrt*  $x * x^{-1}) x$

*sqrt\_mono\_thm*  
 $\vdash \forall x y \bullet 0. \leq x \wedge x < y \Rightarrow$  *Sqrt*  $x <$  *Sqrt*  $y$

*square\_mono\_≤\_thm*  
 $\vdash \forall x y \bullet 0. \leq x \wedge 0. \leq y \Rightarrow (x \leq y \Leftrightarrow x^2 \leq y^2)$

*id\_over\_sqrt\_thm*  
 $\vdash \forall x \bullet 0. < x \Rightarrow x *$  *Sqrt*  $x^{-1} =$  *Sqrt*  $x$

*sqrt\_1\_minus\_sin\_squared\_thm*  
 $\vdash \forall x \bullet$  *Sqrt*  $(1. -$  *Sin*  $x^2) =$  *Abs*  $($  *Cos*  $x)$

*sqrt\_1\_minus\_cos\_squared\_thm*  
 $\vdash \forall x \bullet$  *Sqrt*  $(1. -$  *Cos*  $x^2) =$  *Abs*  $($  *Sin*  $x)$

*tan\_deriv\_thm*  
 $\vdash \forall x \bullet \neg$  *Cos*  $x = 0. \Rightarrow ($  *Tan* *Deriv*  $1. +$  *Tan*  $x^2) x$

*cotan\_deriv\_thm*  
 $\vdash \forall x$   
•  $\neg$  *Sin*  $x = 0. \Rightarrow ($  *Cotan* *Deriv*  $\sim (1. +$  *Cotan*  $x^2)) x$

*sec\_deriv\_thm*  
 $\vdash \forall x \bullet \neg$  *Cos*  $x = 0. \Rightarrow ($  *Sec* *Deriv* *Sec*  $x *$  *Tan*  $x) x$

*cosec\_deriv\_thm*  
 $\vdash \forall x$

$\bullet \neg \text{Sin } x = 0.$   
 $\Rightarrow (\text{Cosec Deriv} \sim (\text{Cosec } x) * \text{Cotan } x) x$

**cos\_0\_less\_thm**  
 $\vdash \forall x \bullet \sim (1 / 2) * \pi < x \wedge x < 1 / 2 * \pi \Rightarrow 0. < \text{Cos } x$

**ArcSin\_consistent**  
 $\vdash \text{Consistent}$   
 $(\lambda \text{ArcSin}'$   
 $\bullet (\forall x \bullet \text{Abs } x \leq 1 / 2 * \pi \Rightarrow \text{ArcSin}' (\text{Sin } x) = x)$   
 $\wedge (\forall x$   
 $\bullet \text{Abs } x \leq 1.$   
 $\Rightarrow \text{Sin } (\text{ArcSin}' x) = x \wedge \text{ArcSin}' \text{Cts } x))$

**arc\_sin\_def**  
 $\vdash (\forall x \bullet \text{Abs } x \leq 1 / 2 * \pi \Rightarrow \text{ArcSin } (\text{Sin } x) = x)$   
 $\wedge (\forall x$   
 $\bullet \text{Abs } x \leq 1. \Rightarrow \text{Sin } (\text{ArcSin } x) = x \wedge \text{ArcSin } \text{Cts } x)$

**sin\_arc\_sin\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow \text{Sin } (\text{ArcSin } x) = x$

**arc\_sin\_sin\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1 / 2 * \pi \Rightarrow \text{ArcSin } (\text{Sin } x) = x$

**arc\_sin\_cts\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow \text{ArcSin } \text{Cts } x$

**abs\_arc\_sin\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow \text{Abs } (\text{ArcSin } x) \leq 1 / 2 * \pi$

**cos\_arc\_sin\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow \text{Cos } (\text{ArcSin } x) = \text{Sqrt } (1. - x \wedge 2)$

**arc\_sin\_1\_minus\_1\_thm**  
 $\vdash \text{ArcSin } 1. = 1 / 2 * \pi \wedge \text{ArcSin } (\sim 1.) = \sim (1 / 2) * \pi$

**arc\_sin\_deriv\_thm**  
 $\vdash \forall x$   
 $\bullet \text{Abs } x < 1.$   
 $\Rightarrow (\text{ArcSin Deriv Sqrt } (1. - x \wedge 2)^{-1}) x$

**ArcCos\_consistent**  
 $\vdash \text{Consistent}$   
 $(\lambda \text{ArcCos}'$   
 $\bullet (\forall x \bullet 0. \leq x \wedge x \leq \pi \Rightarrow \text{ArcCos}' (\text{Cos } x) = x)$   
 $\wedge (\forall x$   
 $\bullet \text{Abs } x \leq 1.$   
 $\Rightarrow \text{Cos } (\text{ArcCos}' x) = x \wedge \text{ArcCos}' \text{Cts } x))$

**arc\_cos\_def**  
 $\vdash (\forall x \bullet 0. \leq x \wedge x \leq \pi \Rightarrow \text{ArcCos } (\text{Cos } x) = x)$   
 $\wedge (\forall x$   
 $\bullet \text{Abs } x \leq 1. \Rightarrow \text{Cos } (\text{ArcCos } x) = x \wedge \text{ArcCos } \text{Cts } x)$

**cos\_arc\_cos\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow \text{Cos } (\text{ArcCos } x) = x$

**arc\_cos\_cos\_thm**  
 $\vdash \forall x \bullet 0. \leq x \wedge x \leq \pi \Rightarrow \text{ArcCos } (\text{Cos } x) = x$

**arc\_cos\_cts\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow \text{ArcCos } \text{Cts } x$

**abs\_arc\_cos\_thm**  
 $\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow 0. \leq \text{ArcCos } x \wedge \text{ArcCos } x \leq \pi$

**sinh\_deriv\_thm**  
 $\vdash \forall x \bullet (\text{Sinh Deriv Cosh } x) x$

**cosh\_deriv\_thm**



$\vdash \forall x \bullet (\text{Cosh Deriv Sinh } x) x$   
**cosh\_0\_less\_thm**  
 $\vdash \forall x \bullet 0. < \text{Cosh } x$   
**cosh\_non\_0\_thm**  
 $\vdash \forall x \bullet \neg \text{Cosh } x = 0.$   
**sinh\_non\_0\_thm**  
 $\vdash \forall x \bullet \text{Sinh } x = 0. \Leftrightarrow x = 0.$   
**cosh\_squared\_minus\_sinh\_squared\_thm**  
 $\vdash \forall x \bullet \text{Cosh } x ^ 2 - \text{Sinh } x ^ 2 = 1.$   
**tanh\_deriv\_thm**  
 $\vdash \forall x \bullet (\text{Tanh Deriv } 1. - \text{Tanh } x ^ 2) x$   
**cotanh\_deriv\_thm**  
 $\vdash \forall x \bullet \neg x = 0. \Rightarrow (\text{Cotanh Deriv } 1. - \text{Cotanh } x ^ 2) x$   
**sech\_deriv\_thm**  
 $\vdash \forall x \bullet (\text{Sech Deriv } \sim (\text{Sech } x) * \text{Tanh } x) x$   
**cosech\_deriv\_thm**  
 $\vdash \forall x$   
 $\bullet \neg x = 0. \Rightarrow (\text{Cosech Deriv } \sim (\text{Cosech } x) * \text{Cotanh } x) x$   
**sinh\_cts\_thm**  $\vdash \forall x \bullet \text{Sinh Cts } x$   
**cosh\_cts\_thm**  $\vdash \forall x \bullet \text{Cosh Cts } x$   
**sinh\_mono\_thm**  
 $\vdash \forall x y \bullet x < y \Rightarrow \text{Sinh } x < \text{Sinh } y$   
**sinh\_cosh\_0\_thm**  
 $\vdash \text{Sinh } 0. = 0. \wedge \text{Cosh } 0. = 1.$   
**sinh\_cosh\_minus\_thm**  
 $\vdash \forall x \bullet \text{Sinh } (\sim x) = \sim (\text{Sinh } x) \wedge \text{Cosh } (\sim x) = \text{Cosh } x$   
**ArcSinh\_consistent**  
 $\vdash \text{Consistent}$   
 $(\lambda \text{ ArcSinh}'$   
 $\bullet \forall x$   
 $\bullet \text{ArcSinh}' (\text{Sinh } x) = x$   
 $\wedge \text{Sinh } (\text{ArcSinh}' x) = x$   
 $\wedge \text{ArcSinh}' \text{ Cts } x)$   
**arc\_sinh\_def**  $\vdash \forall x$   
 $\bullet \text{ArcSinh } (\text{Sinh } x) = x$   
 $\wedge \text{Sinh } (\text{ArcSinh } x) = x$   
 $\wedge \text{ArcSinh } \text{ Cts } x$   
**sqrt\_1\_plus\_sinh\_squared\_thm**  
 $\vdash \forall x \bullet \text{Sqrt } (1. + \text{Sinh } x ^ 2) = \text{Cosh } x$   
**cosh\_arc\_sinh\_thm**  
 $\vdash \forall x \bullet \text{Cosh } (\text{ArcSinh } x) = \text{Sqrt } (1. + x ^ 2)$   
**arc\_sinh\_deriv\_thm**  
 $\vdash \forall x \bullet (\text{ArcSinh Deriv Sqrt } (1. + x ^ 2) ^{-1}) x$   
**gauge\_∩\_thm**  $\vdash \forall G1 G2$   
 $\bullet G1 \in \text{Gauge} \wedge G2 \in \text{Gauge}$   
 $\Rightarrow (\lambda x \bullet G1 x \cap G2 x) \in \text{Gauge}$   
**gauge\_o\_minus\_thm**  
 $\vdash \forall G \bullet G \in \text{Gauge} \Rightarrow (\lambda x \bullet \{t | \sim t \in G (\sim x)\}) \in \text{Gauge}$   
**gauge\_o\_plus\_thm**  
 $\vdash \forall G h$   
 $\bullet G \in \text{Gauge} \Rightarrow (\lambda x \bullet \{t | t + h \in G (x + h)\}) \in \text{Gauge}$

***gauge\_o\_times\_thm***

- $$\vdash \forall c G$$
- $G \in \text{Gauge} \wedge \neg c = 0.$   
 $\Rightarrow (\lambda x \bullet \{t | c * t \in G (c * x)\}) \in \text{Gauge}$

***gauge\_refinement\_thm***

- $$\vdash \forall G1 G2$$
- $G1 \in \text{Gauge} \wedge G2 \in \text{Gauge}$   
 $\Rightarrow (\exists G$ 
    - $G \in \text{Gauge}$   
 $\wedge (\forall x \bullet G x \subseteq G1 x)$   
 $\wedge (\forall x \bullet G x \subseteq G2 x))$

***gauge\_refine\_3\_thm***

- $$\vdash \forall G1 G2 G3$$
- $G1 \in \text{Gauge} \wedge G2 \in \text{Gauge} \wedge G3 \in \text{Gauge}$   
 $\Rightarrow (\exists G$ 
    - $G \in \text{Gauge}$   
 $\wedge (\forall x \bullet G x \subseteq G1 x)$   
 $\wedge (\forall x \bullet G x \subseteq G2 x)$   
 $\wedge (\forall x \bullet G x \subseteq G3 x))$

***fine\_refinement\_thm***

- $$\vdash \forall G1 G2 t I n$$
- $(\forall x \bullet G1 x \subseteq G2 x) \wedge (t, I, n) \in G1 \text{ Fine}$   
 $\Rightarrow (t, I, n) \in G2 \text{ Fine}$

***chosen\_tag\_thm***

- $$\vdash \forall a$$
- $\exists G1$ 
    - $G1 \in \text{Gauge}$   
 $\wedge (\forall G2$ 
      - $G2 \in \text{Gauge} \wedge (\forall x \bullet G2 x \subseteq G1 x)$   
 $\Rightarrow (\forall t I n m$ 
        - $m < n$   
 $\wedge (t, I, n) \in \text{TaggedPartition}$   
 $\wedge (t, I, n) \in G2 \text{ Fine}$   
 $\wedge a$   
 $\in \text{ClosedInterval } (I m) (I (m + 1))$   
 $\Rightarrow t m = a))$

***chosen\_tags\_thm***

- $$\vdash \forall list$$
- $\exists G1$ 
    - $G1 \in \text{Gauge}$   
 $\wedge (\forall G2$ 
      - $G2 \in \text{Gauge} \wedge (\forall x \bullet G2 x \subseteq G1 x)$   
 $\Rightarrow (\forall t I n m a$ 
        - $m < n$   
 $\wedge (t, I, n) \in \text{TaggedPartition}$   
 $\wedge (t, I, n) \in G2 \text{ Fine}$   
 $\wedge a$   
 $\in \text{ClosedInterval } (I m) (I (m + 1))$   
 $\wedge a \in \text{Elems list}$   
 $\Rightarrow t m = a))$

***riemann\_gauge\_thm***

$\vdash \forall e$   
 •  $0. < e$   
 $\Rightarrow (\exists G$   
 •  $G \in Gauge$   
 $\wedge (\forall t I n m$   
 •  $m < n$   
 $\wedge (t, I, n) \in TaggedPartition$   
 $\wedge (t, I, n) \in G Fine$   
 $\Rightarrow I (m + 1) - I m < e))$

**tagged\_partition\_∃\_thm**

**cousin\_lemma**  $\vdash \forall G a b$

•  $G \in Gauge \wedge a \leq b$   
 $\Rightarrow (\exists t I n$   
 •  $I 0 = a$   
 $\wedge I n = b$   
 $\wedge (t, I, n) \in TaggedPartition$   
 $\wedge (t, I, n) \in G Fine)$

**riemann\_sum\_induction\_thm**

$\vdash \forall f t I n$

•  $RiemannSum f (t, I, 0) = 0.$   
 $\wedge RiemannSum f (t, I, n + 1)$   
 $= RiemannSum f (t, I, n)$   
 $+ f (t n) * (I (n + 1) - I n)$

**series\_induction\_thm1**

$\vdash \forall s n$

•  $Sigma s (n + 1) = s 0 + Sigma (\lambda m \bullet s (m + 1)) n$

**ℝ\_abs\_series\_thm**

$\vdash \forall s n \bullet Abs (Sigma s n) \leq Sigma (\lambda m \bullet Abs (s m)) n$

**riemann\_sum\_induction\_thm1**

$\vdash \forall f t I n$

•  $RiemannSum f (t, I, 0) = 0.$   
 $\wedge RiemannSum f (t, I, n + 1)$   
 $= f (t 0) * (I 1 - I 0)$   
 $+ RiemannSum$   
 $f$   
 $((\lambda m \bullet t (m + 1)), (\lambda m \bullet I (m + 1)), n)$

**riemann\_sum\_plus\_thm**

$\vdash \forall f g t I n$

•  $RiemannSum (\lambda x \bullet f x + g x) (t, I, n)$   
 $= RiemannSum f (t, I, n) + RiemannSum g (t, I, n)$

**riemann\_sum\_const\_times\_thm**

$\vdash \forall f c t I n$

•  $RiemannSum (\lambda x \bullet c * f x) (t, I, n)$   
 $= c * RiemannSum f (t, I, n)$

**riemann\_sum\_minus\_thm**

$\vdash \forall f t I n$

•  $RiemannSum (\lambda x \bullet \sim (f x)) (t, I, n)$   
 $= \sim (RiemannSum f (t, I, n))$

**riemann\_sum\_local\_thm**

$\vdash \forall f t I s J n$

•  $(\forall m \bullet m < n \Rightarrow t m = s m) \wedge (\forall m \bullet m \leq n \Rightarrow I m = J m)$

$$\Rightarrow \text{RiemannSum } f (t, I, n) = \text{RiemannSum } f (s, J, n)$$

**riemann\_sum\_o\_minus\_thm**

$$\begin{aligned} &\vdash \forall f t I n \\ &\bullet \text{RiemannSum } f (t, I, n) \\ &= \text{RiemannSum} \\ &\quad (\lambda x \bullet f (\sim x)) \\ &\quad ((\lambda m \bullet \sim (t (n - 1 - m))), \\ &\quad (\lambda m \bullet \sim (I (n - m))), n) \end{aligned}$$

**riemann\_sum\_o\_times\_thm**

$$\begin{aligned} &\vdash \forall f t I n c \\ &\bullet \neg c = 0. \\ &\Rightarrow \text{RiemannSum } (\lambda x \bullet f (c * x)) (t, I, n) \\ &= c^{-1} \\ &\quad * \text{RiemannSum} \\ &\quad f \\ &\quad ((\lambda m \bullet c * t m), (\lambda m \bullet c * I m), n) \end{aligned}$$

**partition\_reverse\_clauses**

$$\begin{aligned} &\vdash \forall f t I n G \\ &\bullet ((t, I, n) \in \text{TaggedPartition} \\ &\quad \Rightarrow ((\lambda m \bullet \sim (t (n - 1 - m))), \\ &\quad (\lambda m \bullet \sim (I (n - m))), n) \\ &\quad \in \text{TaggedPartition}) \\ &\wedge ((t, I, n) \in G \text{ Fine} \\ &\quad \Rightarrow ((\lambda m \bullet \sim (t (n - 1 - m))), \\ &\quad (\lambda m \bullet \sim (I (n - m))), n) \\ &\quad \in (\lambda x \bullet \{t | \sim t \in G (\sim x)\}) \text{ Fine}) \end{aligned}$$

**tagged\_partition\_append\_thm**

$$\begin{aligned} &\vdash \forall s J m t I n \\ &\bullet (s, J, m) \in \text{TaggedPartition} \\ &\quad \wedge (t, I, n) \in \text{TaggedPartition} \\ &\quad \wedge J m = I 0 \\ &\Rightarrow ((\lambda k \bullet \text{if } k < m \text{ then } s k \text{ else } t (k - m)), \\ &\quad (\lambda k \bullet \text{if } k \leq m \text{ then } J k \text{ else } I (k - m)), \\ &\quad m + n) \\ &\in \text{TaggedPartition} \end{aligned}$$

**fine\_append\_thm**

$$\begin{aligned} &\vdash \forall G s J m t I n \\ &\bullet (s, J, m) \in G \text{ Fine} \wedge (t, I, n) \in G \text{ Fine} \wedge J m = I 0 \\ &\Rightarrow ((\lambda k \bullet \text{if } k < m \text{ then } s k \text{ else } t (k - m)), \\ &\quad (\lambda k \bullet \text{if } k \leq m \text{ then } J k \text{ else } I (k - m)), \\ &\quad m + n) \\ &\in G \text{ Fine} \end{aligned}$$

**riemann\_sum\_0\_thm**

$$\vdash \forall t I n \bullet \text{RiemannSum } (\lambda x \bullet 0.) (t, I, n) = 0.$$

**riemann\_sum\_append\_thm**

$$\begin{aligned} &\vdash \forall f t I m n \\ &\bullet \text{RiemannSum } f (t, I, m + n) \\ &= \text{RiemannSum } f (t, I, m) \\ &\quad + \text{RiemannSum} \\ &\quad f \\ &\quad ((\lambda k \bullet t (m + k)), (\lambda k \bullet I (m + k)), n) \end{aligned}$$

*riemann\_sum\_¬\_0\_thm*

- $$\begin{aligned} &\vdash \forall f t I n \\ &\bullet \neg \text{RiemannSum } f (t, I, n) = 0. \\ &\quad \Rightarrow (\exists m \\ &\quad \bullet m < n \\ &\quad \quad \wedge \neg f (t m) = 0. \\ &\quad \quad \wedge \text{RiemannSum } f (t, I, n) \\ &\quad \quad = \text{RiemannSum } f (t, I, m + 1)) \end{aligned}$$

*partition\_mono\_thm*

- $$\begin{aligned} &\vdash \forall t I n k m \\ &\bullet (t, I, n) \in \text{TaggedPartition} \wedge k \leq m \wedge m < n \\ &\quad \Rightarrow I k < I (m + 1) \end{aligned}$$

*tag\_mono\_thm*

- $$\begin{aligned} &\vdash \forall t I n k m \\ &\bullet (t, I, n) \in \text{TaggedPartition} \wedge k \leq m \wedge m < n \\ &\quad \Rightarrow t k \leq t m \end{aligned}$$

*tag\_upper\_bound\_thm*

- $$\begin{aligned} &\vdash \forall t I n k m \\ &\bullet (t, I, n) \in \text{TaggedPartition} \wedge k \leq m \wedge m < n \\ &\quad \Rightarrow t k \leq I (m + 1) \end{aligned}$$

*tag\_lower\_bound\_thm*

- $$\begin{aligned} &\vdash \forall t I n k m \\ &\bullet (t, I, n) \in \text{TaggedPartition} \wedge k \leq m \wedge m < n \\ &\quad \Rightarrow I k \leq t m \end{aligned}$$

*subpartition\_thm*

- $$\begin{aligned} &\vdash \forall n m t I \\ &\bullet m \leq n \wedge (t, I, n) \in \text{TaggedPartition} \\ &\quad \Rightarrow (t, I, m) \in \text{TaggedPartition} \end{aligned}$$

*subpartition\_fine\_thm*

- $$\begin{aligned} &\vdash \forall n m t I n G \\ &\bullet m \leq n \wedge (t, I, n) \in G \text{ Fine} \Rightarrow (t, I, m) \in G \text{ Fine} \end{aligned}$$

*riemann\_sum\_local\_thm1*

- $$\begin{aligned} &\vdash \forall f g t I n \\ &\bullet (\forall x \bullet I 0 \leq x \wedge x \leq I n \Rightarrow f x = g x) \\ &\quad \wedge (t, I, n) \in \text{TaggedPartition} \\ &\quad \Rightarrow \text{RiemannSum } f (t, I, n) = \text{RiemannSum } g (t, I, n) \end{aligned}$$

*riemann\_sum\_0\_≤\_thm*

- $$\begin{aligned} &\vdash \forall f t I n \\ &\bullet (\forall x \bullet 0. \leq f x) \wedge (t, I, n) \in \text{TaggedPartition} \\ &\quad \Rightarrow 0. \leq \text{RiemannSum } f (t, I, n) \end{aligned}$$

*tag\_unique\_thm*

- $$\begin{aligned} &\vdash \forall t I n k m \\ &\bullet (t, I, n) \in \text{TaggedPartition} \\ &\quad \wedge k < m \\ &\quad \wedge m < n \\ &\quad \wedge t k = t m \\ &\quad \Rightarrow m = k + 1 \end{aligned}$$

*partition\_cover\_thm*

- $$\begin{aligned} &\vdash \forall t I n x \\ &\bullet (t, I, n) \in \text{TaggedPartition} \wedge I 0 < x \wedge x \leq I n \\ &\quad \Rightarrow (\exists m \bullet m < n \wedge I m < x \wedge x \leq I (m + 1)) \end{aligned}$$

*partition\_cover\_thm1*

$$\begin{aligned} & \vdash \forall t I n x \\ & \bullet (t, I, n) \in \text{TaggedPartition} \wedge I 0 \leq x \wedge x < I n \\ & \Rightarrow (\exists m \bullet m < n \wedge I m \leq x \wedge x < I (m + 1)) \end{aligned}$$

*riemann\_sum\_χ\_singleton\_thm*

$$\begin{aligned} & \vdash \forall c t I n \\ & \bullet (t, I, n) \in \text{TaggedPartition} \\ & \Rightarrow \text{RiemannSum } (\chi \{c\}) (t, I, n) = 0. \\ & \vee (\exists m \\ & \bullet m < n \\ & \wedge \text{RiemannSum } (\chi \{c\}) (t, I, n) \\ & = I (m + 1) - I m) \\ & \vee (\exists m \\ & \bullet m + 1 < n \\ & \wedge \text{RiemannSum } (\chi \{c\}) (t, I, n) \\ & = I (m + 2) - I m) \end{aligned}$$

*int\_ℝ\_plus\_thm*

$$\begin{aligned} & \vdash \forall f g c d \\ & \bullet f \text{Int}_R c \wedge g \text{Int}_R d \\ & \Rightarrow (\lambda x \bullet f x + g x) \text{Int}_R c + d \end{aligned}$$

*int\_ℝ\_minus\_thm*

$$\vdash \forall f c \bullet f \text{Int}_R c \Rightarrow (\lambda x \bullet \sim (f x)) \text{Int}_R \sim c$$

*int\_ℝ\_0\_thm*  $\vdash (\lambda x \bullet 0.) \text{Int}_R 0.$

*int\_ℝ\_const\_times\_thm*

$$\vdash \forall f c d \bullet f \text{Int}_R d \Rightarrow (\lambda x \bullet c * f x) \text{Int}_R c * d$$

*int\_ℝ\_diff\_0\_thm*

$$\begin{aligned} & \vdash \forall f g c \\ & \bullet (\lambda x \bullet f x - g x) \text{Int}_R 0. \Rightarrow (f \text{Int}_R c \Leftrightarrow g \text{Int}_R c) \end{aligned}$$

*int\_ℝ\_unique\_thm*

$$\vdash \forall f c d \bullet f \text{Int}_R c \wedge f \text{Int}_R d \Rightarrow c = d$$

*int\_ℝ\_0\_≤\_thm*

$$\vdash \forall f c \bullet (\forall x \bullet 0. \leq f x) \wedge f \text{Int}_R c \Rightarrow 0. \leq c$$

*int\_ℝ\_≤\_thm*

$$\vdash \forall f g c d \bullet (\forall x \bullet f x \leq g x) \wedge f \text{Int}_R c \wedge g \text{Int}_R d \Rightarrow c \leq d$$

*int\_ℝ\_0\_dominated\_thm*

$$\begin{aligned} & \vdash \forall f g \\ & \bullet (\forall x \bullet 0. \leq f x) \wedge (\forall x \bullet f x \leq g x) \wedge g \text{Int}_R 0. \\ & \Rightarrow f \text{Int}_R 0. \end{aligned}$$

*int\_ℝ\_χ\_singleton\_lemma*

$$\begin{aligned} & \vdash \forall G e c t I n \\ & \bullet 0. < e \\ & \wedge G \in \text{Gauge} \\ & \wedge (\forall t I n m \\ & \bullet m < n \\ & \wedge (t, I, n) \in \text{TaggedPartition} \\ & \wedge (t, I, n) \in G \text{Fine} \\ & \Rightarrow I (m + 1) - I m < e) \\ & \wedge (t, I, n) \in \text{TaggedPartition} \\ & \wedge (t, I, n) \in G \text{Fine} \\ & \Rightarrow \text{Abs } (\text{RiemannSum } (\chi \{c\}) (t, I, n)) < 2. * e \end{aligned}$$

*int\_ℝ\_χ\_singleton\_thm*

$$\vdash \forall c \bullet \chi \{c\} \text{Int}_R 0.$$

*int\_ℝ\_singleton\_support\_thm*

$$\vdash \forall f c \bullet (\forall x \bullet \neg f x = 0. \Rightarrow x = c) \Rightarrow f \text{ Int}_R 0.$$

*int\_ℝ\_finite\_support\_thm*

$$\vdash \forall f \text{ list}$$

$$\bullet (\forall x \bullet \neg f x = 0. \Rightarrow x \in \text{Elems list}) \Rightarrow f \text{ Int}_R 0.$$

*int\_ℝ\_χ-0-⊆-thm*

$$\vdash \forall A B \bullet A \subseteq B \wedge \chi B \text{ Int}_R 0. \Rightarrow \chi A \text{ Int}_R 0.$$

*int\_ℝ\_χ-0-∩-thm*

$$\vdash \forall A B \bullet \chi A \text{ Int}_R 0. \Rightarrow \chi (A \cap B) \text{ Int}_R 0.$$

*χ-∪-∩-thm*

$$\vdash \forall A B x \bullet \chi (A \cup B) x = \chi A x + \chi B x - \chi (A \cap B) x$$

*int\_ℝ\_χ-0-∪-thm*

$$\vdash \forall A B$$

$$\bullet \chi A \text{ Int}_R 0. \wedge \chi B \text{ Int}_R 0. \Rightarrow \chi (A \cup B) \text{ Int}_R 0.$$

*int\_half\_infinite\_interval\_thm*

$$\vdash \forall f c a \bullet (f \text{ Int } c) \{x|a \leq x\} \Leftrightarrow (f \text{ Int } c) \{x|a < x\}$$

*chosen\_value\_thm*

$$\vdash \forall y f c A$$

$$\bullet (f \text{ Int } c) A$$

$$\Leftrightarrow ((\lambda x \bullet \text{if } x = y \text{ then } 0. \text{ else } f x) \text{ Int } c) A$$

*int\_interval\_thm*

$$\vdash \forall f g a b c$$

$$\bullet (f \text{ Int } c) (\text{ClosedInterval } a b)$$

$$\Leftrightarrow (f \text{ Int } c) (\text{OpenInterval } a b)$$

*int\_ℝ\_o\_minus\_thm*

$$\vdash \forall f c \bullet f \text{ Int}_R c \Leftrightarrow (\lambda x \bullet f (\sim x)) \text{ Int}_R c$$

*int\_ℝ\_o\_plus\_thm*

$$\vdash \forall f c h \bullet f \text{ Int}_R c \Leftrightarrow (\lambda x \bullet f (x + h)) \text{ Int}_R c$$

*int\_ℝ\_o\_times\_thm*

$$\vdash \forall f c d$$

$$\bullet \neg 0. = d \wedge f \text{ Int}_R c$$

$$\Rightarrow (\lambda x \bullet f (d * x)) \text{ Int}_R \text{Abs } d^{-1} * c$$

*int\_ℝ\_support\_bounded\_below\_thm*

$$\vdash \forall a c f$$

$$\bullet (\forall x \bullet x \leq a \Rightarrow f x = 0.)$$

$$\Rightarrow (f \text{ Int}_R c$$

$$\Leftrightarrow (\forall e$$

$$\bullet 0. < e$$

$$\Rightarrow (\exists G b$$

$$\bullet G \in \text{Gauge}$$

$$\wedge a < b$$

$$\wedge (\forall t I n$$

$$\bullet (t, I, n) \in \text{TaggedPartition}$$

$$\wedge I 0 = a$$

$$\wedge b < I n$$

$$\wedge (t, I, n) \in G \text{ Fine}$$

$$\Rightarrow \text{Abs } (\text{RiemannSum } f (t, I, n) - c)$$

$$< e))))))$$

*int\_support\_bounded\_below\_lemma*

$$\vdash \forall a f g c$$

$$\bullet (\forall x \bullet a < x \Rightarrow f x = g x)$$

$$\Rightarrow ((\forall e$$

- $0. < e$ 
  - $\Rightarrow (\exists G b$ 
    - $G \in Gauge$ 
      - $\wedge a < b$
      - $\wedge (\forall t I n$ 
        - $(t, I, n) \in TaggedPartition$ 
          - $\wedge I 0 = a$
          - $\wedge b < I n$
          - $\wedge (t, I, n) \in G Fine$
          - $\Rightarrow Abs (RiemannSum f (t, I, n) - c)$
          - $< e)))$

$\Leftrightarrow (\forall e$

- $0. < e$ 
  - $\Rightarrow (\exists G b$ 
    - $G \in Gauge$ 
      - $\wedge a < b$
      - $\wedge (\forall t I n$ 
        - $(t, I, n) \in TaggedPartition$ 
          - $\wedge I 0 = a$
          - $\wedge b < I n$
          - $\wedge (t, I, n) \in G Fine$
          - $\Rightarrow Abs (RiemannSum g (t, I, n) - c)$
          - $< e))))$

*int\_bounded\_below\_thm*

$\vdash \forall a b c f$

- $(f Int c) \{x | a \leq x\}$ 
  - $\Leftrightarrow (\forall e$ 
    - $0. < e$ 
      - $\Rightarrow (\exists G b$ 
        - $G \in Gauge$ 
          - $\wedge a < b$
          - $\wedge (\forall t I n$ 
            - $(t, I, n) \in TaggedPartition$ 
              - $\wedge I 0 = a$
              - $\wedge b < I n$
              - $\wedge (t, I, n) \in G Fine$
              - $\Rightarrow Abs (RiemannSum f (t, I, n) - c)$
              - $< e)))$

*int\_ℝ\_bounded\_support\_thm*

$\vdash \forall a b c f$

- $a < b$ 
  - $\wedge (\forall x \bullet x \leq a \Rightarrow f x = 0.)$
  - $\wedge (\forall x \bullet b \leq x \Rightarrow f x = 0.)$

$\Rightarrow (f Int_R c$

- $\Leftrightarrow (\forall e$ 
  - $0. < e$ 
    - $\Rightarrow (\exists G$ 
      - $G \in Gauge$ 
        - $\wedge (\forall t I n$ 
          - $(t, I, n) \in TaggedPartition$ 
            - $\wedge I 0 = a$



$$\begin{aligned}
& \wedge I\ n = b \\
& \wedge (t, I, n) \in G\ Fine \\
\Rightarrow & Abs\ (RiemannSum\ f\ (t, I, n) - c) \\
& (< e)))))
\end{aligned}$$

*bounded\_int\_thm*

$$\begin{aligned}
& \vdash \forall a\ b\ c\ f \\
& \bullet a < b \\
& \Rightarrow ((f\ Int\ c)\ (ClosedInterval\ a\ b)) \\
& \Leftrightarrow (\forall e \\
& \bullet 0. < e \\
& \Rightarrow (\exists G \\
& \bullet G \in Gauge \\
& \wedge (\forall t\ I\ n \\
& \bullet (t, I, n) \in TaggedPartition \\
& \wedge I\ 0 = a \\
& \wedge I\ n = b \\
& \wedge (t, I, n) \in G\ Fine \\
& \Rightarrow Abs\ (RiemannSum\ f\ (t, I, n) - c) \\
& (< e)))))
\end{aligned}$$

*bounded\_int\_local\_thm*

$$\begin{aligned}
& \vdash \forall a\ b\ f\ g\ c \\
& \bullet a < b \wedge (\forall x \bullet x \in ClosedInterval\ a\ b \Rightarrow f\ x = g\ x) \\
& \Rightarrow ((g\ Int\ c)\ (ClosedInterval\ a\ b)) \\
& \Leftrightarrow (f\ Int\ c)\ (ClosedInterval\ a\ b))
\end{aligned}$$

*straddle\_thm*

$$\begin{aligned}
& \vdash \forall f\ df\ x\ e \\
& \bullet (f\ Deriv\ df\ x)\ x \wedge 0. < e \\
& \Rightarrow (\exists d \\
& \bullet 0. < d \\
& \wedge (\forall u\ v \\
& \bullet u \in OpenInterval\ (x - d)\ (x + d) \\
& \wedge v \in OpenInterval\ (x - d)\ (x + d) \\
& \wedge u \leq x \\
& \wedge x \leq v \\
& \Rightarrow Abs\ (df\ x * (v - u) - (f\ v - f\ u)) \\
& \leq e * (v - u)))
\end{aligned}$$

*straddle\_gauge\_thm*

$$\begin{aligned}
& \vdash \forall A\ e\ f\ df \\
& \bullet (\forall x \bullet x \in A \Rightarrow (f\ Deriv\ df\ x)\ x) \wedge 0. < e \\
& \Rightarrow (\exists G \\
& \bullet G \in Gauge \\
& \wedge (\forall t\ I\ n\ m \\
& \bullet (t, I, n) \in TaggedPartition \\
& \wedge (t, I, n) \in G\ Fine \\
& \wedge m < n \\
& \wedge t\ m \in A \\
& \Rightarrow Abs \\
& (df\ (t\ m) * (I\ (m + 1) - I\ m) \\
& - (f\ (I\ (m + 1)) - f\ (I\ m))) \\
& \leq e * (I\ (m + 1) - I\ m)))
\end{aligned}$$

*int\_deriv\_thm2*

$$\vdash \forall a\ b\ sf\ f$$

- $a < b$   
 $\wedge (\forall x \bullet a \leq x \wedge x < b \Rightarrow (sf \text{ Deriv } f \ x) \ x)$   
 $\wedge sf \text{ Cts } b$   
 $\Rightarrow (f \text{ Int } sf \ b - sf \ a) \ (\text{ClosedInterval } a \ b)$

**int\_deriv\_thm3**

- $\vdash \forall a \ b \ sf \ f$
- $a < b$   
 $\wedge (\forall x \bullet a < x \wedge x \leq b \Rightarrow (sf \text{ Deriv } f \ x) \ x)$   
 $\wedge sf \text{ Cts } a$   
 $\Rightarrow (f \text{ Int } sf \ b - sf \ a) \ (\text{ClosedInterval } a \ b)$

**int\_deriv\_thm**

- $\vdash \forall a \ b \ sf \ f$
- $a < b$   
 $\wedge (\forall x \bullet a < x \wedge x < b \Rightarrow (sf \text{ Deriv } f \ x) \ x)$   
 $\wedge sf \text{ Cts } a$   
 $\wedge sf \text{ Cts } b$   
 $\Rightarrow (f \text{ Int } sf \ b - sf \ a) \ (\text{ClosedInterval } a \ b)$

**int\_deriv\_thm1**

- $\vdash \forall a \ b \ sf \ f$
- $a < b$   
 $\wedge (\forall x$   
  - $x \in \text{ClosedInterval } a \ b \Rightarrow (sf \text{ Deriv } f \ x) \ x)$ $\Rightarrow (f \text{ Int } sf \ b - sf \ a) \ (\text{ClosedInterval } a \ b)$

**int\_ℝ\_χ\_interval\_thm**

- $\vdash \forall a \ b \bullet a \leq b \Rightarrow \chi \ (\text{ClosedInterval } a \ b) \ \text{Int}_R \ b - a$

**int\_example\_thm**

- $\vdash ((\lambda x \bullet \text{Sqrt } (1. - x \wedge 2) \ ^{-1}) \ \text{Int } \pi)$   
 $(\text{ClosedInterval } (\sim 1.) \ 1.)$

**int\_recip\_thm**

- $\vdash \forall a \ b$
- $0. < a \wedge a < b$   
 $\Rightarrow ((\lambda x \bullet x \ ^{-1}) \ \text{Int } \text{Log } b - \text{Log } a)$   
 $(\text{ClosedInterval } a \ b)$

**log\_minus\_log\_estimate\_thm**

- $\vdash \forall a \ b$
- $0. < a \wedge a < b \Rightarrow \text{Log } b - \text{Log } a \leq (b - a) * a \ ^{-1}$

**harmonic\_series\_estimate\_thm**

- $\vdash \forall m$
- $\text{Log } (\mathbb{N}R \ (m + 1)) \leq \text{Sigma } (\lambda m \bullet \mathbb{N}R \ (m + 1) \ ^{-1}) \ m$

**harmonic\_series\_divergent\_thm**

- $\vdash \forall c \bullet \neg \text{Sigma } (\lambda m \bullet \mathbb{N}R \ (m + 1) \ ^{-1}) \ \rightarrow c$

**area\_unique\_thm**

- $\vdash \forall A \ c \ d \bullet A \ \text{Area } c \wedge A \ \text{Area } d \Rightarrow c = d$

**area\_translate\_thm**

- $\vdash \forall A \ c \ u \ v$
- $A \ \text{Area } c \Rightarrow \{(x, y) | (x + u, y + v) \in A\} \ \text{Area } c$

**area\_dilate\_thm**

- $\vdash \forall A \ c \ d \ e$
- $A \ \text{Area } c \wedge \neg d = 0. \wedge \neg e = 0.$   
 $\Rightarrow \{(x, y) | (d \ ^{-1} * x, e \ ^{-1} * y) \in A\}$   
 $\text{Area } \text{Abs } d * \text{Abs } e * c$

**area\_dilate\_thm1**

$\vdash \forall A c d$   
 •  $A \text{ Area } c \wedge \neg d = 0.$   
 $\Rightarrow \{(x, y) | (d^{-1} * x, d^{-1} * y) \in A\}$   
 $\text{Area } d^{\wedge 2} * c$

**area\_empty\_thm**

$\vdash \{\} \text{Area } 0.$

**area\_Union\_thm**

$\vdash \forall A B c d y$   
 •  $A \text{ Area } c \wedge B \text{ Area } d \wedge A \cap B \text{ Area } y$   
 $\Rightarrow A \cup B \text{ Area } c + d - y$

**area\_Intersection\_thm**

$\vdash \forall A B c d y$   
 •  $A \text{ Area } c \wedge B \text{ Area } d \wedge A \cup B \text{ Area } y$   
 $\Rightarrow A \cap B \text{ Area } c + d - y$

**area\_rectangle\_thm**

$\vdash \forall w h$   
 •  $0. \leq h \wedge 0. \leq w$   
 $\Rightarrow \{(x, y) | x \in \text{ClosedInterval } 0. w$   
 $\wedge y \in \text{ClosedInterval } 0. h\}$   
 $\text{Area } w * h$

**area\_circle\_lemma1**

$\vdash \forall x$   
 •  $\text{Abs } x < 1.$   
 $\Rightarrow ((\lambda x \bullet x * \text{Sqrt } (1. - x^{\wedge 2}) + \text{ArcSin } x)$   
 $\text{Deriv } 2. * \text{Sqrt } (1. - x^{\wedge 2}))$   
 $x$

**area\_circle\_lemma2**

$\vdash \forall x \bullet \text{Abs } x \leq 1. \Rightarrow 0. \leq 1. - x^{\wedge 2}$

**area\_circle\_lemma3**

$\vdash \forall x$   
 •  $\text{Abs } x \leq 1.$   
 $\Rightarrow (\lambda x \bullet x * \text{Sqrt } (1. - x^{\wedge 2}) + \text{ArcSin } x) \text{ Cts } x$

**area\_circle\_lemma4**

$\vdash \forall r x y$   
 •  $\text{Abs } x \leq r$   
 $\Rightarrow (\text{Sqrt } (x^{\wedge 2} + y^{\wedge 2}) \leq r$   
 $\Leftrightarrow \text{Abs } y \leq \text{Sqrt } (r^{\wedge 2} - x^{\wedge 2}))$

**area\_circle\_int\_thm**

$\vdash ((\lambda x \bullet 2. * \text{Sqrt } (1. - x^{\wedge 2})) \text{ Int } \pi)$   
 $(\text{ClosedInterval } (\sim 1.) 1.)$

**area\_unit\_circle\_thm**

$\vdash \{(x, y) | \text{Sqrt } (x^{\wedge 2} + y^{\wedge 2}) \leq 1.\} \text{Area } \pi$

**circle\_dilate\_thm**

$\vdash \forall r$   
 •  $0. < r$   
 $\Rightarrow \{(x, y) | \text{Sqrt } (x^{\wedge 2} + y^{\wedge 2}) \leq r\}$   
 $= \{(x, y) | \text{Sqrt } ((r^{-1} * x)^{\wedge 2} + (r^{-1} * y)^{\wedge 2})$   
 $\leq 1.\}$

**area\_circle\_thm**

$\vdash \forall r$

- $0. < r$   
 $\Rightarrow \{(x, y) | \text{Sqrt } (x^2 + y^2) \leq r\}$   
 $\text{Area } \pi * r^2$

**buffon\_needle\_lemma**

$\vdash \text{let } S$   
 $= \{(\theta, d)$   
 $|\theta \in \text{ClosedInterval } 0. \pi$   
 $\wedge d \in \text{ClosedInterval } 0. 1.\}$   
 $\text{in let } x\_axis = \{(x, y) | y = 0.\}$   
 $\text{in let needle } (\theta, d)$   
 $= \{(x, y)$   
 $|\exists t$   

- $t \in \text{ClosedInterval } 0. 1.$   
 $\wedge x = t * \text{Cos } \theta$   
 $\wedge y = d - t * \text{Sin } \theta\}$

 $\text{in let } X$   
 $= \{(\theta, d)$   
 $|\{(\theta, d) \in S$   
 $\wedge \neg \text{needle } (\theta, d) \cap x\_axis = \{\}\}$   
 $\text{in } X \text{ Area } 2.$

**buffon\_needle\_thm**

$\vdash \text{let } S$   
 $= \{(\theta, d)$   
 $|\theta \in \text{ClosedInterval } 0. \pi$   
 $\wedge d \in \text{ClosedInterval } 0. 1.\}$   
 $\text{in let } x\_axis = \{(x, y) | y = 0.\}$   
 $\text{in let needle } (\theta, d)$   
 $= \{(x, y)$   
 $|\exists t$   

- $t \in \text{ClosedInterval } 0. 1.$   
 $\wedge x = t * \text{Cos } \theta$   
 $\wedge y = d - t * \text{Sin } \theta\}$

 $\text{in let } X$   
 $= \{(\theta, d)$   
 $|\{(\theta, d) \in S$   
 $\wedge \neg \text{needle } (\theta, d) \cap x\_axis = \{\}\}$   
 $\text{in } X \subseteq S$   
 $\wedge (\exists x s$   

- $\neg s = 0.$   
 $\wedge X \text{ Area } x$   
 $\wedge S \text{ Area } s$   
 $\wedge x / s = 2. / \pi)$

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