Mathematical Case Studies: the Complex Numbers*

Rob Arthan
rda@lemma-one.com

12 October 2017

Abstract

Definitions of the complex numbers and their arithmetic operators in ProofPower-HOL with proofs of some of their elementary algebraic properties.

Reference: LEMMA1/HOL/WRK072; Current git revision: b6fbf81

*First posted 17 September 2005; for full changes history see: https://github.com/RobArthan/pp-contrib.

Contents

1 INTRODUCTION **3**

2 THE ALGEBRA ℂ **3**

2.1 Preliminaries . . . 3

2.2 The Type ℂ . . . 3

2.3 Algebraic Operators . . . 4

2.3.1 Addition . . . 4

2.3.2 Negation . . . 4

2.3.3 Multiplication . . . 4

2.3.4 Embedding of the Real Numbers . . . 5

2.3.5 The Imaginary Unit . . . 5

2.3.6 Conjugation . . . 5

2.3.7 Reciprocal . . . 5

2.3.8 Subtraction . . . 5

2.3.9 Division . . . 6

2.3.10 Embedding of the Natural Numbers . . . 6

2.3.11 Exponentiation with Natural Number Exponents . . . 6

2.4 ℂ *qua* Real Normed Space . . . 7

2.5 Transcendental Functions . . . 7

2.5.1 Exponential . . . 7

2.5.2 Group Structures . . . 7

3 POLYNOMIALS **8**

4 TOPOLOGICAL ASPECTS **9**

A THEOREMS IN THE THEORY ℂ **11**

INDEX **22**

1 INTRODUCTION

This document contains the beginnings of a theory of complex arithmetic in ProofPower-HOL.

After some preliminaries, section 2 introduces the field operations and the embedddings of the real numbers and of conjugation (which help to abbreviate the definition of reciprocals). We then define the derived operations of binary subtraction and division and the embedding of the natural numbers.

In section 4 we introduce the topology on the complex numbers and define the exponential mapping of the real line onto the unit circle.

A proof script contained in the source of this document but hidden from the printed document proves several theorems about the complex numbers. A listing of the theorems proved is given in appendix A.

We prove that the complex numbers form a field and prove some elementary algebraic properties of complex conjugation and of the embeddings of the natural numbers and the real numbers in the complex numbers. The proofs of these basic algebraic properties generally comprise little more than simplifying using definitions and standard theorems and then application of the decision procedure for linear arithmetic to prove the resulting problem of real algebra. One exception is proving that $zz^{-1} = 1$, which requires a few lines of reasoning about real squares.

The proof scripts then go on to develop basic facts about the exponential mapping of the real line onto the unit circle, leading up to the important fact that it is a covering projection.

2 THE ALGEBRA ℂ

2.1 Preliminaries

The following commands set up a theory to hold the definitions, theorems, etc. The theory has the theory of analysis defined in [1] for its parent, although the elementary algebraic theory only needs one or two little theorems about squares.

SML

```
force_delete_theory "ℂ" handle Fail _ => ();
open_theory "analysis";
new_theory "ℂ";
new_parent "homotopy";
new_parent "group_egs";
```

Now set up a convenient proof context:

SML

```
set_merge_pcs["basic_hol1", "'ℤ", "'ℝ", "'sets_alg"];
```

2.2 The Type ℂ

The type ℂ comprises pairs of real numbers. We capture this in a type abbreviation:

SML

```
declare_type_abbrev("ℂ", [], ⌜:ℝ × ℝ⌝);
```

We declare aliases for the instances of the projection functions that give the real and imaginary part of a complex number:

SML

```
app declare_alias [
        ("Re", ⌜Fst : ℂ → ℝ⌝),
        ("Im", ⌜Snd : ℂ → ℝ⌝)];
```

2.3 Algebraic Operators

2.3.1 Addition

SML

```
declare_infix(300, "+C");
```

HOL Constant

$\$+_C : \mathbb{C} \to \mathbb{C} \to \mathbb{C}$

$\forall\ z\ w : \mathbb{C} \bullet\ z +_C w = (Re\ z + Re\ w, Im\ z + Im\ w)$

SML

```
declare_alias("+", ⌜$+C⌝);
```

2.3.2 Negation

HOL Constant

$\sim_C : \mathbb{C} \to \mathbb{C}$

$\forall\ z : \mathbb{C} \bullet\ \sim_C z = (\sim(Re\ z), \sim(Im\ z))$

SML

```
declare_alias("~", ⌜~C⌝);
```

2.3.3 Multiplication

SML

```
declare_infix(310, "*C");
```

HOL Constant

$\$*_C : \mathbb{C} \to \mathbb{C} \to \mathbb{C}$

$\forall\ z\ w : \mathbb{C} \bullet$
$\quad z *_C w = (Re\ z * Re\ w - Im\ z * Im\ w, Re\ z * Im\ w + Im\ z * Re\ w)$

SML

```
declare_alias("*", ⌜$*C⌝);
```

2.3.4 Embedding of the Real Numbers

HOL Constant

$\mathbb{RC} : \mathbb{R} \to \mathbb{C}$

$\forall x \bullet\ \mathbb{RC}\ x = (x,\ \mathbb{NR}\ 0)$

HOL Constant

$\mathbb{R}\iota : \mathbb{R} \to \mathbb{C}$

$\forall x \bullet\ \mathbb{R}\iota\ x = (\mathbb{NR}\ 0,\ x)$

2.3.5 The Imaginary Unit

HOL Constant

$\boldsymbol{I_C} : \mathbb{C}$

$I_C = \mathbb{R}\iota\ 1.0$

2.3.6 Conjugation

SML

```
declare_postfix(320, "‾");
```

HOL Constant

$\$^{-} : \mathbb{C} \to \mathbb{C}$

$\forall\ z : \mathbb{C} \bullet\ z\ ^{-} = (Re\ z,\ \sim(Im\ z))$

2.3.7 Reciprocal

SML

```
declare_postfix(320, "⁻¹C");
```

HOL Constant

$\$^{-1C} : \mathbb{C} \to \mathbb{C}$

$\forall\ z : \mathbb{C} \bullet\ z\ ^{-1\,C} = z\ ^{-} * \mathbb{RC}(Re\ (z * z\ ^{-})\ ^{-1})$

SML

```
declare_alias("⁻¹", ⌜$⁻¹C⌝);
```

2.3.8 Subtraction

SML

```
declare_infix(300, "−C");
```

HOL Constant

$\$-_{\boldsymbol{C}} : \mathbb{C} \to \mathbb{C} \to \mathbb{C}$

$\forall\ z\ w : \mathbb{C}\bullet\ z\ -_C\ w = z + \sim w$

2.3.9 Division

SML

```
declare_infix(315, "/_C");
```

HOL Constant

$\$/_{\boldsymbol{C}} : \mathbb{C} \to \mathbb{C} \to \mathbb{C}$

$\forall\ z\ w : \mathbb{C}\bullet\ z\ /_C\ w = z * (w\ ^{-1})$

SML

```
declare_alias("/", ⌜$/_C⌝);
```

2.3.10 Embedding of the Natural Numbers

HOL Constant

$\mathbb{N}\mathbb{C} : \mathbb{N} \to \mathbb{C}$

$\forall m\bullet\ \mathbb{N}\mathbb{C}\ m = \mathbb{R}\mathbb{C}(\mathbb{N}\mathbb{R}\ m)$

HOL Constant

$\mathbb{N}\boldsymbol{\iota} : \mathbb{N} \to \mathbb{C}$

$\forall m\bullet\ \mathbb{N}\iota\ m = (0., \mathbb{N}\mathbb{R}\ m)$

2.3.11 Exponentiation with Natural Number Exponents

SML

```
declare_infix(320, "^_C");
```

HOL Constant

$\$\hat{}_{\boldsymbol{C}} : \mathbb{C} \to \mathbb{N} \to \mathbb{C}$

$(\forall\ z{:}\ \mathbb{C}\bullet\ z\ \hat{}_C\ 0 = \mathbb{N}\mathbb{C}\ 1)$

$\wedge\quad (\forall\ z{:}\ \mathbb{C};\ m\bullet\ z\ \hat{}_C\ (m{+}1) = z * z\ \hat{}_C\ m)$

SML

```
declare_alias("^", ⌜$^_C⌝);
```

2.4 $\mathbb{C}$ *qua* Real Normed Space

SML

```
declare_infix(310, "*_RC");
```

HOL Constant

$\$*_{\boldsymbol{RC}} : \mathbb{R} \to \mathbb{C} \to \mathbb{C}$

$\forall\ x{:}\ \mathbb{R};\ v : \mathbb{C}\bullet$
$\quad x *_{RC} v = \mathbb{RC}\ x *_C v$

HOL Constant

$\boldsymbol{Abs_C} : \mathbb{C} \to \mathbb{R}$

$\forall\ v : \mathbb{C}\bullet$
$\quad Abs_C\ v = Sqrt(Re\ (v * v\ ^{-}))$

SML

```
declare_alias("Abs", ⌜Abs_C⌝);
```

2.5 Transcendental Functions

2.5.1 Exponential

HOL Constant

$\boldsymbol{Exp_C} : \mathbb{C} \to \mathbb{C}$

$\forall z\bullet\ Exp_C\ z = \mathbb{RC}\ (Exp(Re\ z)) * (Cos\ (Im\ z),\ Sin\ (Im\ z))$

SML

```
declare_alias("Exp", ⌜$Exp_C⌝);
```

2.5.2 Group Structures

We now define the additive and multiplicative groups of complex numbers.

HOL Constant

$\mathbb{C}_+ :\ \mathbb{C}\ GROUP;$
$\mathbb{C}_* :\ \mathbb{C}\ GROUP$

$\quad \mathbb{C}_+ = MkGROUP\ Universe\ \$+\ (\mathbb{NC}\ 0)\ \sim$
$\wedge \quad \mathbb{C}_* = MkGROUP\ \{x \mid \neg x = \mathbb{NC}\ 0\}\ \$*\ (\mathbb{NC}\ 1)\ \$^{-1}$

3 POLYNOMIALS

As we did for the real numbers in [1], we define the set of polynomial function on the complex numbers to be the smallest set of functions that contains all constant functions and the identity function and that is closed under pointwise addition and multiplication of functions.

HOL Constant

$$\boldsymbol{PolyFunc_C} : (\mathbb{C} \to \mathbb{C})\ SET$$

$$
\begin{array}{lll}
PolyFunc_C = \bigcap & & \\
\quad \{ & A & \\
\quad | & (\forall c\bullet\ (\lambda x\bullet c) \in A) & \\
\quad \wedge & (\lambda x\bullet x) \in A & \\
\quad \wedge & (\forall f\ g\bullet f \in A \wedge g \in A \Rightarrow (\lambda x\bullet f\ x + g\ x) \in A) & \\
\quad \wedge & (\forall f\ g\bullet f \in A \wedge g \in A \Rightarrow (\lambda x\bullet f\ x * g\ x) \in A)\ \} &
\end{array}
$$

The following function gives the n-th partial sum of a series.

HOL Constant

$$\boldsymbol{Sigma_C} : (\mathbb{N} \to \mathbb{C}) \to (\mathbb{N} \to \mathbb{C})$$

$$
\begin{array}{ll}
 & (\forall s\bullet\ Sigma_C\ s\ 0 = \mathbb{N}\mathbb{C}\ 0) \\
\wedge & (\forall s\ n\bullet\ Sigma_C\ s\ (n+1) = Sigma_C\ s\ n + s\ n)
\end{array}
$$

We represent a complex polynomial as a pair (s, n) where s is a sequence of coefficients and n is a bound on the degree. The following function maps such a pair to the polynomial function it represents.

HOL Constant

$$\boldsymbol{Poly_C} : (\mathbb{N} \to \mathbb{C}) \times \mathbb{N} \to \mathbb{C} \to \mathbb{C}$$

$$\forall c\ n\ z\bullet\ Poly_C\ (c,\ n)\ z = Sigma_C\ (\lambda i\bullet\ c\ i * z\hat{\ }i)\ (n+1)$$

We now give the operations on coefficients that correspond to addition of polynomial functions ...

HOL Constant

$$\boldsymbol{PlusCoeffs_C} : ((\mathbb{N} \to \mathbb{C}) \times \mathbb{N}) \to ((\mathbb{N} \to \mathbb{C}) \times \mathbb{N}) \to ((\mathbb{N} \to \mathbb{C}) \times \mathbb{N})$$

$$
\begin{array}{l}
\forall s\ m\ t\ n\bullet \\
\quad PlusCoeffs_C\ (s,\ m)\ (t,\ n) = \\
\quad ((\lambda i\bullet\ \ (if\ i \le m\ then\ s\ i\ else\ \mathbb{N}\mathbb{C}\ 0) + \\
\quad\quad\quad (if\ i \le n\ then\ t\ i\ else\ \mathbb{N}\mathbb{C}\ 0)),\ m+n)
\end{array}
$$

... and to multiplication of one polynomial function by another.

HOL Constant

$\boldsymbol{TimesCoeffs_C} : ((\mathbb{N} \to \mathbb{C}) \times \mathbb{N}) \to ((\mathbb{N} \to \mathbb{C}) \times \mathbb{N}) \to ((\mathbb{N} \to \mathbb{C}) \times \mathbb{N})$

$$(\forall s\ t\ n\bullet\ TimesCoeffs_C\ (s,\ 0)\ (t,\ n) = ((\lambda i\bullet\ s\ 0 * t\ i),\ n))$$
$$\land\quad (\forall s\ m\ t\ n\bullet\ TimesCoeffs_C\ (s,\ m{+}1)\ (t,\ n) = PlusCoeffs_C\ (TimesCoeffs_C\ (s,\ m)\ (t,\ n))\ ((\lambda i\bullet\ if\ i \le m\ then\ \mathbb{N}\mathbb{C}\ 0\ else\ s\ (m{+}1) * t\ (i-(m{+}1))),\ m{+}n{+}1))$$

4 TOPOLOGICAL ASPECTS

The standard topology on $\mathbb{C}$ is just the product topology:

HOL Constant

$\boldsymbol{Open_C} : \mathbb{C}\ SET\ SET$

$$Open_C = O_R \times_T O_R$$

As with $Open_R$ it is convenient to have a short name for the topology:

SML

```
declare_alias("O_C", ⌜Open_C⌝);
```

The unit circle S^1:

HOL Constant

$\boldsymbol{S1} : \mathbb{C}\ SET$

$$S1 = \{z \mid Abs_C\ z = 1.\}$$

The topology on the unit circle:

HOL Constant

$\boldsymbol{Open_{S1}} : \mathbb{C}\ SET\ SET$

$$Open_{S1} = S1 \lhd_T O_C$$

Again it is convenient to have a short name for the topology:

SML

```
declare_alias("O_S1", ⌜Open_S1⌝);
```

and the exponential mapping of the real line onto the unit circle:

HOL Constant

$\boldsymbol{ExpS1} : \mathbb{R} \to \mathbb{C}$

$$\forall t\bullet\ ExpS1\ t = (Cos\ t,\ Sin\ t)$$

The degree of a loop in the unit circle:

HOL Constant

$\boldsymbol{LoopS1Degree} : (\mathbb{R} \to \mathbb{C}) \to \mathbb{Z}$

$\forall x\ f\ g \bullet$
$\quad f \in Loops(O_{S1},\ x) \land g \in Paths\ O_R \land (\forall s \bullet\ ExpS1(g\ s) = f\ s)$
$\Rightarrow \quad g\ 1. - g\ 0. = 2. * \mathbb{ZR}\ (LoopS1Degree\ f) * \pi$

The generator of the fundamental group of the circle.

HOL Constant

$\boldsymbol{IotaS1} : \mathbb{R} \to \mathbb{C}$

$IotaS1 = (\lambda t \bullet\ if\ t \le 0. \lor 1. \le t\ then\ \mathbb{NC}\ 1\ else\ ExpS1(2. * \pi * t))$

The function that converts a self-mapping of the unit circle into a loop.

HOL Constant

$\boldsymbol{S1S1Loop} : (\mathbb{C} \to \mathbb{C}) \to (\mathbb{R} \to \mathbb{C})$

$\forall f \bullet \quad S1S1Loop\ f = (\lambda t \bullet\ f\ (IotaS1\ t))$

The degree of a self-mapping of the unit circle:

HOL Constant

$\boldsymbol{S1S1Degree} : (\mathbb{C} \to \mathbb{C}) \to \mathbb{Z}$

$\forall f \bullet \quad S1S1Degree\ f = LoopS1Degree\ (S1S1Loop\ f)$

The degree of an element of the fundamental group of the unit circle:

HOL Constant

$\boldsymbol{ClassS1Degree} : (\mathbb{R} \to \mathbb{C})\ \mathbb{P} \to \mathbb{Z}$

$\forall x\ p\ f \bullet$
$\quad p \in Loops(O_{S1},\ x)\ /\ PathHomotopic\ O_{S1} \land f \in p$
$\Rightarrow \quad ClassS1Degree\ p = LoopS1Degree\ f$

References

[1] LEMMA1/HOL/WRK066. *Mathematical Case Studies: Basic Analysis.* R.D. Arthan, Lemma 1 Ltd., rda@lemma-one.com.

A THEOREMS IN THE THEORY $\mathbb{C}$

$\mathbb{C}$_plus_comm_thm

$\vdash \forall\ x\ y \bullet\ x + y = y + x$

$\mathbb{C}$_plus_assoc_thm

$\vdash \forall\ x\ y\ z \bullet\ (x + y) + z = x + y + z$

$\mathbb{C}$_plus_assoc_thm1

$\vdash \forall\ x\ y\ z \bullet\ x + y + z = (x + y) + z$

$\mathbb{C}$_plus_0_thm $\vdash \forall\ x \bullet\ x + \mathbb{NC}\ 0 = x \land \mathbb{NC}\ 0 + x = x$

$\mathbb{C}$_plus_minus_thm

$\vdash \forall\ x \bullet\ x + \sim x = \mathbb{NC}\ 0 \land \sim x + x = \mathbb{NC}\ 0$

$\mathbb{C}$_times_conj_thm

$\vdash \forall\ x \bullet\ x * x\ \bar{}\ = \mathbb{RC}\ (Re\ x\ \hat{}\ 2 + Im\ x\ \hat{}\ 2)$

$\mathbb{C}$_times_comm_thm

$\vdash \forall\ x\ y \bullet\ x * y = y * x$

$\mathbb{C}$_times_assoc_thm

$\vdash \forall\ x\ y\ z \bullet\ (x * y) * z = x * y * z$

$\mathbb{C}$_times_assoc_thm1

$\vdash \forall\ x\ y\ z \bullet\ x * y * z = (x * y) * z$

$\mathbb{C}$_times_1_thm

$\vdash \forall\ x \bullet\ x * \mathbb{NC}\ 1 = x \land \mathbb{NC}\ 1 * x = x$

$\mathbb{R}$_square_plus_square_eq_0_thm

$\vdash \forall\ x\ y \bullet\ x\ \hat{}\ 2 + y\ \hat{}\ 2 = 0. \Leftrightarrow x = 0. \land y = 0.$

$\mathbb{C}$_times_recip_thm

$\vdash \forall\ x \bullet\ \neg\ x = \mathbb{NC}\ 0 \Rightarrow x * x^{-1} = \mathbb{NC}\ 1$

$\mathbb{C}$_times_recip_thm1

$\vdash \forall\ x \bullet\ \neg\ x = \mathbb{NC}\ 0 \Rightarrow x^{-1} * x = \mathbb{NC}\ 1$

$\mathbb{C}$_times_eq_1_thm

$\vdash \forall\ z\ w \bullet\ \neg\ z = \mathbb{NC}\ 0 \land z * w = \mathbb{NC}\ 1 \Rightarrow w = z^{-1}$

$\mathbb{C}$_times_plus_distrib_thm

$\vdash \forall\ x\ y\ z$
$\bullet\ x * (y + z) = x * y + x * z$
$\land\ (x + y) * z = x * z + y * z$

$\mathbb{C}$_conj_plus_homomorphism_thm

$\vdash \forall\ x\ y \bullet\ (x + y)\ \bar{}\ = x\ \bar{}\ + y\ \bar{}$

$\mathbb{C}$_conj_times_homomorphism_thm

$\vdash \forall\ x\ y \bullet\ (x * y)\ \bar{}\ = x\ \bar{}\ * y\ \bar{}$

$\mathbb{RC}$_plus_homomorphism_thm

$\vdash \forall\ x\ y \bullet\ \mathbb{RC}\ (x + y) = \mathbb{RC}\ x + \mathbb{RC}\ y$

$\mathbb{RC}$_times_homomorphism_thm

$\vdash \forall\ x\ y \bullet\ \mathbb{RC}\ (x * y) = \mathbb{RC}\ x * \mathbb{RC}\ y$

$\mathbb{NC}$_plus_homomorphism_thm

$\vdash \forall\ x\ y \bullet\ \mathbb{NC}\ (x + y) = \mathbb{NC}\ x + \mathbb{NC}\ y$

$\mathbb{NC}$_one_one_thm

$\vdash \forall\ m\ n \bullet\ \mathbb{NC}\ m = \mathbb{NC}\ n \Leftrightarrow m = n$

$\mathbb{NC}$_times_homomorphism_thm

$\vdash \forall\ x\ y \bullet\ \mathbb{NC}\ (x * y) = \mathbb{NC}\ x * \mathbb{NC}\ y$

$\mathbb{C}$_plus_order_thm

$\vdash \forall\ x\ y\ z$
$\bullet\ y + x = x + y$
$\land\ (x + y) + z = x + y + z$

$\wedge\ y + x + z = x + y + z$

ℂ_eq_thm $\vdash \forall\ x\ y \bullet\ x = y \Leftrightarrow x + \sim y = \mathbb{N}\mathbb{C}\ 0$

ℂ_minus_clauses

$\vdash \forall\ x\ y$
$\bullet \sim (\sim x) = x$
$\wedge\ x + \sim x = \mathbb{N}\mathbb{C}\ 0$
$\wedge \sim x + x = \mathbb{N}\mathbb{C}\ 0$
$\wedge \sim (x + y) = \sim x + \sim y$
$\wedge \sim (\mathbb{N}\mathbb{C}\ 0) = \mathbb{N}\mathbb{C}\ 0$

ℂ_plus_clauses

$\vdash \forall\ x\ y\ z$
$\bullet\ (x + z = y + z \Leftrightarrow x = y)$
$\wedge\ (z + x = y + z \Leftrightarrow x = y)$
$\wedge\ (x + z = z + y \Leftrightarrow x = y)$
$\wedge\ (z + x = z + y \Leftrightarrow x = y)$
$\wedge\ (x + z = z \Leftrightarrow x = \mathbb{N}\mathbb{C}\ 0)$
$\wedge\ (z + x = z \Leftrightarrow x = \mathbb{N}\mathbb{C}\ 0)$
$\wedge\ (z = z + y \Leftrightarrow y = \mathbb{N}\mathbb{C}\ 0)$
$\wedge\ (z = y + z \Leftrightarrow y = \mathbb{N}\mathbb{C}\ 0)$
$\wedge\ x + \mathbb{N}\mathbb{C}\ 0 = x$
$\wedge\ \mathbb{N}\mathbb{C}\ 0 + x = x$
$\wedge\ \neg\ \mathbb{N}\mathbb{C}\ 1 = \mathbb{N}\mathbb{C}\ 0$
$\wedge\ \neg\ \mathbb{N}\mathbb{C}\ 0 = \mathbb{N}\mathbb{C}\ 1$

ℂ_times_order_thm

$\vdash \forall\ x\ y\ z$
$\bullet\ y * x = x * y$
$\wedge\ (x * y) * z = x * y * z$
$\wedge\ y * x * z = x * y * z$

ℂ_times_0_thm

$\vdash \forall\ x \bullet\ x * \mathbb{N}\mathbb{C}\ 0 = \mathbb{N}\mathbb{C}\ 0 \wedge \mathbb{N}\mathbb{C}\ 0 * x = \mathbb{N}\mathbb{C}\ 0$

ℂ_times_eq_0_thm

$\vdash \forall\ x\ y \bullet\ x * y = \mathbb{N}\mathbb{C}\ 0 \Rightarrow x = \mathbb{N}\mathbb{C}\ 0 \vee y = \mathbb{N}\mathbb{C}\ 0$

ℂ_times_eq_0_thm1

$\vdash \forall\ x\ y \bullet\ x * y = \mathbb{N}\mathbb{C}\ 0 \Leftrightarrow x = \mathbb{N}\mathbb{C}\ 0 \vee y = \mathbb{N}\mathbb{C}\ 0$

ℂ_times_clauses

$\vdash \forall\ x$
$\bullet\ \mathbb{N}\mathbb{C}\ 0 * x = \mathbb{N}\mathbb{C}\ 0$
$\wedge\ x * \mathbb{N}\mathbb{C}\ 0 = \mathbb{N}\mathbb{C}\ 0$
$\wedge\ x * \mathbb{N}\mathbb{C}\ 1 = x$
$\wedge\ \mathbb{N}\mathbb{C}\ 1 * x = x$

ℂ_times_minus_thm

$\vdash \forall\ x\ y$
$\bullet \sim x * y = \sim (x * y)$
$\wedge\ x * \sim y = \sim (x * y)$
$\wedge \sim x * \sim y = x * y$

ℂ_ℕ_exp_1_thm

$\vdash \forall\ n \bullet\ \mathbb{N}\mathbb{C}\ 1\ \hat{}\ n = \mathbb{N}\mathbb{C}\ 1$

ℂ_ℕ_exp_0_thm

$\vdash \forall\ n \bullet\ \mathbb{N}\mathbb{C}\ 0\ \hat{}\ (n + 1) = \mathbb{N}\mathbb{C}\ 0$

ℂ_ℕ_exp_rw_thm

$\vdash \forall\ z\ n$

$\bullet\ z \mathbin{\hat{}} n = (if\ n = 0\ then\ \mathbb{NC}\ 1\ else\ z * z \mathbin{\hat{}} (n - 1))$

$\mathbb{C}$_$\mathbb{N}$_***exp_plus_thm***

$\vdash \forall\ z\ m\ n \bullet\ z \mathbin{\hat{}} (m + n) = z \mathbin{\hat{}} m * z \mathbin{\hat{}} n$

$\mathbb{C}$_$\mathbb{N}$_***exp_clauses***

$\vdash \forall\ z\ n$
$\bullet\ z \mathbin{\hat{}} 0 = \mathbb{NC}\ 1$
$\land\ z \mathbin{\hat{}} 1 = z$
$\land\ \mathbb{NC}\ 1 \mathbin{\hat{}} n = \mathbb{NC}\ 1$
$\land\ \mathbb{NC}\ 0 \mathbin{\hat{}} (n + 1) = \mathbb{NC}\ 0$

$\mathbb{C}$_$\mathbb{N}$_***exp_times_thm***

$\vdash \forall\ n\ z\ w \bullet\ (z * w) \mathbin{\hat{}} n = z \mathbin{\hat{}} n * w \mathbin{\hat{}} n$

$\mathbb{C}$_$\mathbb{N}$_***exp_¬_eq_0_thm***

$\vdash \forall\ n\ z \bullet\ \neg\ z = \mathbb{NC}\ 0 \Rightarrow \neg\ z \mathbin{\hat{}} n = \mathbb{NC}\ 0$

$\mathbb{C}$_$\mathbb{N}$_***exp_recip_thm***

$\vdash \forall\ n\ z \bullet\ \neg\ z = \mathbb{NC}\ 0 \Rightarrow (z^{\ -1}) \mathbin{\hat{}} n = (z \mathbin{\hat{}} n)^{\ -1}$

$\mathbb{C}$_$\mathbb{N}$_***exp_minus_thm***

$\vdash \forall\ m\ n\ z$
$\bullet\ m \le n \land \neg\ z = \mathbb{NC}\ 0$
$\Rightarrow z \mathbin{\hat{}} (n - m) = z \mathbin{\hat{}} n * (z \mathbin{\hat{}} m)^{\ -1}$

$\mathbb{C}$_***additive_group_thm***

$\vdash \mathbb{C}_+ \in Group$

$\mathbb{C}$_***multiplicative_group_thm***

$\vdash \mathbb{C}_* \in Group$

$\mathbb{C}$_***additive_ops_thm***

$\vdash Car\ \mathbb{C}_+ = Universe$
$\land\ (\forall\ x\ y \bullet\ (x\ .\ y)\ \mathbb{C}_+ = x + y)$
$\land\ Unit\ \mathbb{C}_+ = \mathbb{NC}\ 0$
$\land\ (\forall\ x \bullet\ (x\ ^{\sim})\ \mathbb{C}_+ = \sim x)$

$\mathbb{C}$_***multiplicative_ops_thm***

$\vdash Car\ \mathbb{C}_* = \{x | \neg\ x = \mathbb{NC}\ 0\}$
$\land\ (\forall\ x\ y \bullet\ (x\ .\ y)\ \mathbb{C}_* = x * y)$
$\land\ Unit\ \mathbb{C}_* = \mathbb{NC}\ 1$
$\land\ (\forall\ x \bullet\ (x\ ^{\sim})\ \mathbb{C}_* = x^{\ -1})$

$\mathbb{C}$_***eq***_$\mathbb{R}$_×_$\mathbb{R}$_***thm***

$\vdash \mathbb{C}_+ = \mathbb{R}_+ \times_G \mathbb{R}_+$

$\mathbb{C}$_***additive***_$\mathbb{C}$_***multiplicative_homomorphism_def***

$\vdash \forall\ f$
$\bullet\ f \in Homomorphism\ (\mathbb{C}_+, \mathbb{C}_*)$
$\Leftrightarrow (\forall\ x \bullet\ \neg\ f\ x = \mathbb{NC}\ 0)$
$\land\ (\forall\ x\ y \bullet\ f\ (x + y) = f\ x * f\ y)$

$\mathbb{C}$_***exp_homomorphism_thm***

$\vdash Exp \in Homomorphism\ (\mathbb{C}_+, \mathbb{C}_*)$

$\mathbb{C}$_***linear_homomorphism_thm***

$\vdash \forall\ c \bullet\ (\lambda\ x \bullet\ c * x) \in Homomorphism\ (\mathbb{C}_+, \mathbb{C}_+)$

de_moivre_thm

$\vdash \forall\ x\ m$
$\bullet\ (Cos\ x, Sin\ x) \mathbin{\hat{}} m$
$= (Cos\ (\mathbb{NR}\ m * x), Sin\ (\mathbb{NR}\ m * x))$

$\mathbb{C}$_***sigma_rw_thm***

$\vdash \forall\ s\ n$
$\bullet\ Sigma_C\ s\ n$

$= (\mathit{if}\ n = 0$
$\mathit{then}\ \mathbb{NC}\ 0$
$\mathit{else}\ s\ (n - 1) + \mathit{Sigma}_C\ s\ (n - 1))$

ℂ_poly_rec_thm

$\vdash (\forall\ s\ z \bullet \mathit{Poly}_C\ (s,\ 0)\ z = s\ 0)$
$\wedge (\forall\ s\ n\ z$
$\bullet \mathit{Poly}_C\ (s,\ n + 1)\ z$
$= \mathit{Poly}_C\ (s,\ n)\ z + s\ (n + 1) * z\ \hat{}\ (n + 1))$

ℂ_poly_eq_thm1

$\vdash \forall\ s\ t\ m\ z$
$\bullet (\forall\ i \bullet i \le m \Rightarrow s\ i = t\ i)$
$\Rightarrow \mathit{Poly}_C\ (s,\ m)\ z = \mathit{Poly}_C\ (t,\ m)\ z$

ℂ_poly_eq_thm2

$\vdash \forall\ s\ m\ n\ z$
$\bullet (\forall\ i \bullet m < i \Rightarrow s\ i = \mathbb{NC}\ 0) \wedge m \le n$
$\Rightarrow \mathit{Poly}_C\ (s,\ m)\ z = \mathit{Poly}_C\ (s,\ n)\ z$

ℂ_poly_eq_thm

$\vdash \forall\ s\ t\ m\ n\ z$
$\bullet (\forall\ i \bullet i \le m \Rightarrow s\ i = t\ i)$
$\wedge (\forall\ i \bullet m < i \Rightarrow t\ i = \mathbb{NC}\ 0)$
$\wedge m \le n$
$\Rightarrow \mathit{Poly}_C\ (s,\ m)\ z = \mathit{Poly}_C\ (t,\ n)\ z$

ℂ_poly_0_thm $\vdash \forall\ n\ z \bullet \mathit{Poly}_C\ ((\lambda\ i \bullet \mathbb{NC}\ 0),\ n)\ z = \mathbb{NC}\ 0$

ℂ_poly_minus_thm

$\vdash \forall\ s\ n\ z$
$\bullet \mathit{Poly}_C\ ((\lambda\ i \bullet \sim (s\ i)),\ n)\ z = \sim (\mathit{Poly}_C\ (s,\ n)\ z)$

ℂ_poly_plus_thm

$\vdash \forall\ s\ t\ n\ z$
$\bullet \mathit{Poly}_C\ ((\lambda\ i \bullet s\ i + t\ i),\ n)\ z$
$= \mathit{Poly}_C\ (s,\ n)\ z + \mathit{Poly}_C\ (t,\ n)\ z$

ℂ_const_eval_thm

$\vdash \forall\ c \bullet (\lambda\ x \bullet c) = \mathit{Poly}_C\ ((\lambda\ i \bullet c),\ 0)$

ℂ_id_eval_thm

$\vdash (\lambda\ x \bullet x)$
$= \mathit{Poly}_C\ ((\lambda\ i \bullet \mathit{if}\ i = 1\ \mathit{then}\ \mathbb{NC}\ 1\ \mathit{else}\ \mathbb{NC}\ 0),\ 1)$

ℂ_plus_eval_thm

$\vdash \forall\ s\ m\ t\ n$
$\bullet (\lambda\ x \bullet \mathit{Poly}_C\ (s,\ m)\ x + \mathit{Poly}_C\ (t,\ n)\ x)$
$= \mathit{Poly}_C\ (\mathit{PlusCoeffs}_C\ (s,\ m)\ (t,\ n))$

ℂ_plus_eval_rw_thm

$\vdash \forall\ sm\ tn$
$\bullet \mathit{Poly}_C\ (\mathit{PlusCoeffs}_C\ sm\ tn)$
$= (\lambda\ x \bullet \mathit{Poly}_C\ sm\ x + \mathit{Poly}_C\ tn\ x)$

ℂ_const_times_eval_thm

$\vdash \forall\ c\ s\ m$
$\bullet \mathit{Poly}_C\ ((\lambda\ i \bullet c * s\ i),\ m)$
$= (\lambda\ x \bullet c * \mathit{Poly}_C\ (s,\ m)\ x)$

ℂ_times_eval_thm

$\vdash \forall\ s\ m\ t\ n$
$\bullet (\lambda\ x \bullet \mathit{Poly}_C\ (s,\ m)\ x * \mathit{Poly}_C\ (t,\ n)\ x)$
$= \mathit{Poly}_C\ (\mathit{TimesCoeffs}_C\ (s,\ m)\ (t,\ n))$

$\mathbb{C}$_times_eval_rw_thm

$\vdash \forall\ sm\ tn$
$\bullet\ Poly_C\ (TimesCoeffs_C\ sm\ tn)$
$= (\lambda\ x\bullet\ Poly_C\ sm\ x * Poly_C\ tn\ x)$

$\mathbb{C}$_poly_induction_thm

$\vdash \forall\ p$
$\bullet\ (\forall\ c\bullet\ p\ (\lambda\ x\bullet\ c))$
$\land\ p\ (\lambda\ x\bullet\ x)$
$\land\ (\forall\ f\ g\bullet\ p\ f \land p\ g \Rightarrow p\ (\lambda\ x\bullet\ f\ x + g\ x))$
$\land\ (\forall\ f\ g\bullet\ p\ f \land p\ g \Rightarrow p\ (\lambda\ x\bullet\ f\ x * g\ x))$
$\Rightarrow (\forall\ h\bullet\ h \in PolyFunc_C \Rightarrow p\ h)$

$\mathbb{C}$_poly_func_eq_poly_eval_thm

$\vdash PolyFunc_C = \{f | \exists\ s\ n\bullet\ f = Poly_C\ (s,\ n)\}$

open_$\mathbb{C}$_topology_thm

$\vdash O_C \in Topology$

open_$\mathbb{C}$_hausdorff_thm

$\vdash O_C \in Hausdorff$

universe_open_$\mathbb{C}$_thm

$\vdash Universe \in O_C$

space_t_open_$\mathbb{C}$_thm

$\vdash Space_T\ O_C = Universe$

space_t_subspace_open_$\mathbb{C}$_thm

$\vdash \forall\ A\bullet\ Space_T\ (A \lhd_T O_C) = A$

subspace_$\mathbb{C}$_topology_thm

$\vdash \forall\ X\bullet\ X \lhd_T O_C \in Topology$

$\mathbb{C}$_minus_continuous_thm

$\vdash \sim\ \in (O_C,\ O_C)\ Continuous$

$\mathbb{C}$_conj_continuous_thm

$\vdash \$^{-} \in (O_C,\ O_C)\ Continuous$

$\mathbb{C}$_plus_continuous_thm

$\vdash Uncurry\ \$+ \in (O_C \times_T O_C,\ O_C)\ Continuous$

$\mathbb{C}$_times_continuous_thm

$\vdash Uncurry\ \$* \in (O_C \times_T O_C,\ O_C)\ Continuous$

$\mathbb{C}$_$\mathbb{N}$_exp_continuous_thm

$\vdash \forall\ n\bullet\ (\lambda\ x\bullet\ x\ \hat{}\ n) \in (O_C,\ O_C)\ Continuous$

$\mathbb{C}$_sigma_continuous_thm

$\vdash \forall\ \sigma\ s\ n$
$\bullet\ \sigma \in Topology \land (\forall\ i\bullet\ s\ i \in (\sigma,\ O_C)\ Continuous)$
$\Rightarrow (\lambda\ x\bullet\ Sigma_C\ (\lambda\ i\bullet\ s\ i\ x)\ n)$
$\in (\sigma,\ O_C)\ Continuous$

$\mathbb{C}$_poly_func_continuous_thm

$\vdash \forall\ f\bullet\ f \in PolyFunc_C \Rightarrow f \in (O_C,\ O_C)\ Continuous$

exp_s1_continuous_thm

$\vdash ExpS1 \in (O_R,\ O_C)\ Continuous$

$\mathbb{C}$_abs_squared_lemma

$\vdash \forall\ x\ y\bullet\ Sqrt\ (x\ \hat{}\ 2 + y\ \hat{}\ 2) = 1. \Leftrightarrow x\ \hat{}\ 2 + y\ \hat{}\ 2 = 1.$

$\in$_s1_lemma $\vdash \forall\ z\bullet\ z \in S1 \Leftrightarrow Re\ z\ \hat{}\ 2 + Im\ z\ \hat{}\ 2 = 1.$

open_s1_topology_thm

$\vdash O_{S1} \in Topology$

space_t_open_s1_thm

$\vdash Space_T\ O_{S1} = S1$

open_ℂ_const_continuous_thm
$\vdash \forall\ \sigma\ c \bullet\ \sigma \in Topology \Rightarrow (\lambda\ x \bullet\ c) \in (\sigma,\ O_C)\ Continuous$

open_ℂ_id_continuous_thm
$\vdash (\lambda\ x \bullet\ x) \in (O_C,\ O_C)\ Continuous$

ℝ_0_≤_*square_plus_square_thm*
$\vdash \forall\ x\ y \bullet\ 0. \leq x\ \hat{}\ 2 + y\ \hat{}\ 2$

ℝ_*square_eq_0_thm*
$\vdash \forall\ x \bullet\ 0. \leq x \Rightarrow (Sqrt\ x = 0. \Leftrightarrow x = 0.)$

ℝ_2*d_cauchy_schwarz_thm*
$\vdash \forall\ a\ b\ c\ d$
$\bullet\ a * c + b * d$
$\leq Sqrt\ (a\ \hat{}\ 2 + b\ \hat{}\ 2) * Sqrt\ (c\ \hat{}\ 2 + d\ \hat{}\ 2)$

ℝ_2*d_triangle_inequality_thm*
$\vdash \forall\ a\ b\ c\ d$
$\bullet\ Sqrt\ ((a + c)\ \hat{}\ 2 + (b + d)\ \hat{}\ 2)$
$\leq Sqrt\ (a\ \hat{}\ 2 + b\ \hat{}\ 2) + Sqrt\ (c\ \hat{}\ 2 + d\ \hat{}\ 2)$

ℂ_*abs_times_thm*
$\vdash \forall\ z\ w \bullet\ Abs\ (z * w) = Abs\ z * Abs\ w$

ℂ_*abs_plus_thm*
$\vdash \forall\ z\ w \bullet\ Abs\ (z + w) \leq Abs\ z + Abs\ w$

ℂ_*abs_eq_0_thm*
$\vdash \forall\ z \bullet\ Abs\ z = 0. \Leftrightarrow z = ℕℂ\ 0$

ℂ_*abs_eq_0_thm*1
$\vdash \forall\ z \bullet\ z = ℕℂ\ 0 \Leftrightarrow Abs\ z = 0.$

ℂ_0_≤_*abs_thm*
$\vdash \forall\ z \bullet\ 0. \leq Abs\ z$

ℂ_*abs_continuous_thm*
$\vdash Abs \in (O_C,\ O_R)\ Continuous$

exp_exp_s1_thm
$\vdash ExpS1 = Exp\ o\ ℝι$

exp_s1_homomorphism_thm
$\vdash \forall\ x\ y \bullet\ ExpS1\ (x + y) = ExpS1\ x * ExpS1\ y$

exp_s1_homomorphism_thm1
$\vdash \forall\ x\ y \bullet\ ExpS1\ x * ExpS1\ y = ExpS1\ (x + y)$

exp_s1_minus_thm
$\vdash \forall\ x \bullet\ ExpS1\ (\sim x) = ExpS1\ x^{-1}$

exp_s1_∈_s1_thm
$\vdash \forall\ x \bullet\ ExpS1\ x \in S1$

exp_s1_period_thm
$\vdash \forall\ x\ y$
$\bullet\ ExpS1\ x = ExpS1\ y$
$\Leftrightarrow (\exists\ m$
$\bullet\ y = x + ℕℝ\ (2 * m) * \pi$
$\lor x = y + ℕℝ\ (2 * m) * \pi)$

exp_s1_period_thm1
$\vdash \forall\ x\ y$
$\bullet\ ExpS1\ x = ExpS1\ y \Leftrightarrow (\exists\ i \bullet\ y = x + 2. * ℤℝ\ i * \pi)$

exp_s1_onto_thm
$\vdash \forall\ z$
$\bullet\ z \in S1 \Rightarrow (\exists_1\ x \bullet\ 0. \leq x \land x < 2. * \pi \land z = ExpS1\ x)$

s1_times_thm $\vdash \forall\ z\ w \bullet\ z \in S1 \land w \in S1 \Rightarrow z * w \in S1$

exp_s1_onto_thm1

$\vdash \forall\ c\ z$
$\bullet\ z \in S1$
$\Rightarrow (\exists_1\ x\bullet\ c - \pi \le x \land x < c + \pi \land z = ExpS1\ x)$

exp_s1_onto_thm2

$\vdash \forall\ c\ z$
$\bullet\ z \in S1 \land \neg\ z = ExpS1\ (c + \pi)$
$\Rightarrow (\exists_1\ x$
$\bullet\ x \in OpenInterval\ (c - \pi)\ (c + \pi) \land z = ExpS1\ x)$

exp_s1_onto_thm3

$\vdash \forall\ z\bullet\ z \in S1 \Rightarrow (\exists\ x\bullet\ ExpS1\ x = z)$

exp_s1_covering_projection_lemma1

$\vdash \forall\ c$
$\bullet\ S1 \setminus \{ExpS1\ (c + \pi)\}$
$= \{z$
$|\exists\ x$
$\bullet\ x \in OpenInterval\ (c - \pi)\ (c + \pi)$
$\land z = ExpS1\ x\}$

exp_s1_covering_projection_lemma2

$\vdash \forall\ x$
$\bullet\ ExpS1$
$\in (OpenInterval\ (x - \pi)\ (x + \pi) \lhd_T\ O_R,$
$(S1 \setminus \{ExpS1\ (x + \pi)\}) \lhd_T\ O_C)\ Homeomorphism$

exp_s1_covering_projection_lemma3

$\vdash ExpS1 \in (O_R,\ S1 \lhd_T\ O_C)\ Continuous$

exp_s1_covering_projection_lemma4

$\vdash \forall\ x\bullet\ ExpS1\ x \in S1 \setminus \{ExpS1\ (x + \pi)\}$

exp_s1_covering_projection_lemma5

$\vdash \forall\ x\ y$
$\bullet\ ExpS1\ x = ExpS1\ y$
$\land \neg\ OpenInterval\ (x + \sim \pi)\ (x + \pi)$
$\cap\ OpenInterval\ (y + \sim \pi)\ (y + \pi)$
$= \{\}$
$\Rightarrow x = y$

$\mathbb{C}$***_punctured_set_thm***

$\vdash \forall\ X\ x\bullet\ X \setminus \{x\} \in X \lhd_T\ O_C$

exp_s1_covering_projection_thm

$\vdash ExpS1 \in (O_R,\ O_{S1})\ CoveringProjection$

exp_s1_path_lifting_thm

$\vdash \forall\ y\ f$
$\bullet\ f \in Paths\ O_{S1} \land ExpS1\ y = f\ 0.$
$\Rightarrow (\exists\ g$
$\bullet\ g \in Paths\ O_R$
$\land g\ 0. = y$
$\land (\forall\ s\bullet\ ExpS1\ (g\ s) = f\ s))$

exp_s1_path_lifting_thm1

$\vdash \forall\ y\ f$
$\bullet\ f \in Paths\ O_{S1}$
$\Rightarrow (\exists\ g\bullet\ g \in Paths\ O_R \land (\forall\ s\bullet\ ExpS1\ (g\ s) = f\ s))$

exp_s1_unique_path_lifting_thm

$\vdash \forall\ f\ g\ a$

$\bullet\ f \in Paths\ O_R$
$\quad \land\ g \in Paths\ O_R$
$\quad \land\ (\forall\ x \bullet\ ExpS1\ (f\ x) = ExpS1\ (g\ x))$
$\quad \land\ g\ a = f\ a$
$\Rightarrow (\forall\ x \bullet\ f\ x = g\ x)$

translated_path_ℝ_path_thm

$\vdash \forall\ f\ c \bullet\ f \in Paths\ O_R \Rightarrow (\lambda\ x \bullet\ f\ x + c) \in Paths\ O_R$

ℤℝ_*minus_homomorphism_thm*

$\vdash \forall\ i \bullet\ \mathbb{ZR}\ (\sim i) = \sim (\mathbb{ZR}\ i)$

LoopS1Degree_consistent

$\vdash Consistent$
$(\lambda\ LoopS1Degree'$
$\bullet\ \forall\ x\ f\ g$
$\bullet\ f \in Loops\ (O_{S1},\ x)$
$\quad \land\ g \in Paths\ O_R$
$\quad \land\ (\forall\ s \bullet\ ExpS1\ (g\ s) = f\ s)$
$\Rightarrow g\ 1. - g\ 0.$
$\quad = 2. * \mathbb{ZR}\ (LoopS1Degree'\ f) * \pi)$

exp_s1_path_fibration_thm

$\vdash \forall\ f\ H$
$\bullet\ f \in Paths\ O_R$
$\quad \land\ H \in (O_R \times_T O_R,\ O_{S1})\ Continuous$
$\quad \land\ (\forall\ x \bullet\ H\ (x,\ 0.) = ExpS1\ (f\ x))$
$\Rightarrow (\exists\ L$
$\bullet\ L \in (O_R \times_T O_R,\ O_R)\ Continuous$
$\quad \land\ (\forall\ x \bullet\ L\ (x,\ 0.) = f\ x)$
$\quad \land\ (\forall\ x\ s$
$\quad \bullet\ s \in ClosedInterval\ 0.\ 1.$
$\quad \Rightarrow ExpS1\ (L\ (x,\ s)) = H\ (x,\ s)))$

exp_s1_thm $\vdash ExpS1\ 0. = \mathbb{NC}\ 1$

exp_s1_2_π_thm

$\vdash \forall\ i \bullet\ ExpS1\ (2. * \mathbb{ZR}\ i * \pi) = \mathbb{NC}\ 1$

iota_s1_loop_thm

$\vdash IotaS1 \in Loops\ (O_{S1},\ \mathbb{NC}\ 1)$

iota_s1_continuous_thm

$\vdash IotaS1 \in (O_R,\ O_{S1})\ Continuous$

s1_s1_loop_loop_thm

$\vdash \forall\ f$
$\bullet\ f \in (O_{S1},\ O_{S1})\ Continuous$
$\Rightarrow S1S1Loop\ f \in Loops\ (O_{S1},\ f\ (\mathbb{NC}\ 1))$

π_recip_clauses

$\vdash \pi * \pi^{-1} = 1. \land \pi^{-1} * \pi = 1.$

times_2_π_lemma

$\vdash \forall\ x\ y \bullet\ x = 2. * y * \pi \Leftrightarrow y = (1\ /\ 2) * x * \pi^{-1}$

ℤ_≤_*cases_thm1*

$\vdash \forall\ i\ j \bullet\ i + \mathbb{NZ}\ 1 \leq j \lor i = j \lor j + \mathbb{NZ}\ 1 \leq i$

ℝ_*times_mono_⇔_thm1*

$\vdash \forall\ x \bullet\ 0. < x \Rightarrow (\forall\ y\ z \bullet\ x * y < x * z \Leftrightarrow y < z)$

ℤℝ_0_*less_thm*

$\vdash \forall\ i \bullet\ 0. < \mathbb{ZR}\ i \Leftrightarrow \mathbb{NZ}\ 0 < i$

ℤℝ_*less_thm* $\vdash \forall\ i\ j \bullet\ \mathbb{ZR}\ i < \mathbb{ZR}\ j \Leftrightarrow i < j$

ℤℝ_one_one_thm

$\vdash \forall\ i\ j \bullet\ \mathbb{ZR}\ i = \mathbb{ZR}\ j \Leftrightarrow i = j$

discrete_subgroup_ℝ_discrete_thm

$\vdash \forall\ c$
$\bullet\ 0. < c \Rightarrow \{z | \exists\ i \bullet\ z = c * \mathbb{ZR}\ i\} \lhd_T O_R \in Discrete_T$

ker_exp_s1_discrete_thm

$\vdash \{z | \exists\ i \bullet\ z = 2. * \mathbb{ZR}\ i * \pi\} \lhd_T O_R \in Discrete_T$

s1_s1_degree_homotopy_invariant_lemma1

$\vdash \forall\ H$
$\bullet\ H \in (O_{S1} \times_T O_R, O_{S1})\ Continuous$
$\Rightarrow (\lambda\ (y, s) \bullet\ H\ (IotaS1\ y, s))$
$\in (O_R \times_T O_R, O_{S1})\ Continuous$

s1_s1_degree_homotopy_invariant_lemma2

$\vdash \forall\ H$
$\bullet\ L \in (O_R \times_T O_R, O_R)\ Continuous$
$\Rightarrow (\lambda\ y \bullet\ L\ (IotaI\ y, 1.)) \in Paths\ O_R$

s1_s1_degree_homotopy_invariant_thm

$\vdash \forall\ f\ g$
$\bullet\ ((O_{S1}, \{\}, O_{S1})\ Homotopic)\ f\ g$
$\Rightarrow S1S1Degree\ f = S1S1Degree\ g$

one_∈_s1_thm $\vdash \mathbb{NC}\ 1 \in S1$

ℂ_ℕ_exp_∈_s1_thm

$\vdash \forall\ n\ z \bullet\ z \in S1 \Rightarrow z\ \hat{}\ n \in S1$

ℂ_ℕ_exp_s1_s1_continuous_thm

$\vdash \forall\ n \bullet\ (\lambda\ z \bullet\ z\ \hat{}\ n) \in (O_{S1}, O_{S1})\ Continuous$

s1_s1_degree_const_thm

$\vdash \forall\ n\ w \bullet\ w \in S1 \Rightarrow S1S1Degree\ (\lambda\ z \bullet\ w) = \mathbb{NZ}\ 0$

s1_s1_degree_ℂ_ℕ_exp_thm

$\vdash \forall\ n \bullet\ S1S1Degree\ (\lambda\ z \bullet\ z\ \hat{}\ n) = \mathbb{NZ}\ n$

s1_s1_loop_eq_thm

$\vdash \forall\ f\ g$
$\bullet\ (\forall\ z \bullet\ z \in S1 \Rightarrow f\ z = g\ z) \Rightarrow S1S1Loop\ f = S1S1Loop\ g$

s1_s1_degree_eq_thm

$\vdash \forall\ f\ g$
$\bullet\ (\forall\ z \bullet\ z \in S1 \Rightarrow f\ z = g\ z)$
$\Rightarrow S1S1Degree\ f = S1S1Degree\ g$

ℝ_recip_ℂ_abs_continuous_thm

$\vdash (\lambda\ z \bullet\ Abs\ z\ ^{-1})$
$\in (\sim \{\mathbb{NC}\ 0\} \lhd_T O_C, \{x | 0. < x\} \lhd_T O_R)\ Continuous$

ℂ_recip_¬_eq_0_thm

$\vdash \forall\ z \bullet\ \neg\ z = \mathbb{NC}\ 0 \Rightarrow \neg\ z\ ^{-1} = \mathbb{NC}\ 0$

ℝℂ_thm $\vdash \mathbb{RC} \in (O_R, O_C)\ Continuous$

s1_times_continuous_thm

$\vdash Uncurry\ \$* \in (O_{S1} \times_T O_{S1}, O_{S1})\ Continuous$

ℂ_abs_ℝ_abs_thm

$\vdash \forall\ x \bullet\ Abs\ (\mathbb{RC}\ x) = Abs\ x$

pp_s1_∈_s1_thm

$\vdash \forall\ z \bullet\ \neg\ z = \mathbb{NC}\ 0 \Rightarrow z * \mathbb{RC}\ (Abs\ z\ ^{-1}) \in S1$

ℂ_poly_continuous_thm

$\vdash \forall\ c\ n \bullet\ Poly_C\ (c, n) \in (O_C, O_C)\ Continuous$

pp_s1_continuous_thm

$\vdash (\lambda\ z\bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))$
$\quad \in (\sim \{\mathbb{NC}\ 0\} \lhd_T\ O_C,\ O_{S1})\ Continuous$

ℂ_sigma_times_thm

$\vdash \forall\ z\ c\ n\bullet\ Sigma_C\ (\lambda\ i\bullet\ z * c\ i)\ n = z * Sigma_C\ c\ n$

ℂ_sigma_times_thm1

$\vdash \forall\ z\ c\ n\bullet\ z * Sigma_C\ c\ n = Sigma_C\ (\lambda\ i\bullet\ z * c\ i)\ n$

ℂ_sigma_const_thm

$\vdash \forall\ n\ z\bullet\ Sigma_C\ (\lambda\ i\bullet\ z)\ n = \mathbb{NC}\ n * z$

ℂ_sigma_eq_thm

$\vdash \forall\ n\ c\ d$
$\quad \bullet (\forall\ i\bullet\ i < n \Rightarrow c\ i = d\ i)$
$\qquad \Rightarrow Sigma_C\ c\ n = Sigma_C\ d\ n$

ℂ_poly_times_thm

$\vdash \forall\ c\ n\ w\ z$
$\quad \bullet\ Poly_C\ ((\lambda\ i\bullet\ w * c\ i),\ n)\ z = w * Poly_C\ (c,\ n)\ z$

sqrt_≤_1_thm $\vdash \forall\ x\bullet\ 0. \le x \land x \le 1. \Rightarrow Sqrt\ x \le 1.$

ℂ_poly_degree_0_lemma1

$\vdash \forall\ c\ n$
$\quad \bullet (\forall\ z\bullet\ Abs\ z \le 1. \Rightarrow \neg\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$
$\qquad \Rightarrow (\lambda\ z\bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))$
$\qquad\quad o\ (\lambda\ (z,\ t)$
$\qquad\quad \bullet\ Poly_C\ (c,\ n)\ (\mathbb{RC}\ (IotaI\ t) * z))$
$\qquad\ \in (O_{S1}\ \times_T\ O_R,\ O_{S1})\ Continuous$

ℂ_poly_degree_0_lemma2

$\vdash \forall\ c\ n$
$\quad \bullet (\forall\ z\bullet\ Abs\ z \le 1. \Rightarrow \neg\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$
$\qquad \Rightarrow (\exists\ w$
$\qquad \bullet\ w \in S1$
$\qquad\quad \land ((O_{S1},\ \{\},\ O_{S1})\ Homotopic)$
$\qquad\qquad (\lambda\ z\bullet\ w)$
$\qquad\qquad ((\lambda\ z\bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))$
$\qquad\qquad\quad o\ Poly_C\ (c,\ n)))$

ℂ_poly_degree_n_lemma1

$\vdash \forall\ c\ n\ z\ t$
$\quad \bullet\ Sigma_C$
$\qquad (\lambda\ i\bullet\ c\ i * \mathbb{RC}\ (IotaI\ t) \verb|^| (n - i) * z \verb|^| i)$
$\qquad (n + 1)$
$\qquad = \mathbb{RC}\ (IotaI\ t)$
$\qquad\quad * Sigma_C$
$\qquad\qquad (\lambda\ i$
$\qquad\qquad \bullet\ c\ i$
$\qquad\qquad\quad * \mathbb{RC}\ (IotaI\ t) \verb|^| (n - 1 - i)$
$\qquad\qquad\quad * z \verb|^| i)$
$\qquad\qquad n$
$\qquad + c\ n * z \verb|^| n$

ℂ_poly_degree_n_lemma2

$\vdash \forall\ c\ n\ z\ t$
$\quad \bullet \neg\ IotaI\ t = 0.$
$\qquad \Rightarrow Sigma_C$
$\qquad\quad (\lambda\ i\bullet\ c\ i * \mathbb{RC}\ (IotaI\ t) \verb|^| (n - i) * z \verb|^| i)$
$\qquad\quad (n + 1)$

$= \mathbb{RC}\ (IotaI\ \ t)\ \hat{}\ n$
$*\ Poly_C\ (c,\ n)\ (z * \mathbb{RC}\ (IotaI\ \ t)\ ^{-1})$

ℂ_poly_degree_n_lemma3

$\vdash \forall\ c\ n$
$\bullet\ (\lambda\ (z,\ t)$
$\bullet\ Sigma_C$
$(\lambda\ i \bullet\ c\ i * \mathbb{RC}\ (IotaI\ \ t)\ \hat{}\ (n - i) * z\ \hat{}\ i)$
$(n + 1))$
$\in (O_C \times_T O_R,\ O_C)\ Continuous$

ℂ_poly_degree_n_lemma4

$\vdash \forall\ c\ n$
$\bullet\ (\forall\ z \bullet\ 1. \le Abs\ z \Rightarrow \neg\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$
$\wedge\ c\ n = \mathbb{NC}\ 1$
$\Rightarrow (\lambda\ z \bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))$
$o\ (\lambda\ (z,\ t)$
$\bullet\ Sigma_C$
$(\lambda\ i$
$\bullet\ c\ i * \mathbb{RC}\ (IotaI\ \ t)\ \hat{}\ (n - i) * z\ \hat{}\ i)$
$(n + 1))$
$\in (O_{S1} \times_T O_R,\ O_{S1})\ Continuous$

ℂ_poly_degree_n_lemma5

$\vdash \forall\ c\ n$
$\bullet\ (\forall\ z \bullet\ 1. \le Abs\ z \Rightarrow \neg\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$
$\wedge\ c\ n = \mathbb{NC}\ 1$
$\Rightarrow ((O_{S1},\ \{\},\ O_{S1})\ Homotopic)$
$((\lambda\ z \bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))\ o\ (\lambda\ z \bullet\ z\ \hat{}\ n))$
$((\lambda\ z \bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))\ o\ Poly_C\ (c,\ n))$

ℂ_poly_degree_0_thm

$\vdash \forall\ c\ n$
$\bullet\ (\forall\ z \bullet\ Abs\ z \le 1. \Rightarrow \neg\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$
$\Rightarrow S1S1Degree$
$((\lambda\ z \bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))\ o\ Poly_C\ (c,\ n))$
$= \mathbb{NZ}\ 0$

ℂ_poly_degree_n_thm

$\vdash \forall\ c\ n$
$\bullet\ (\forall\ z \bullet\ 1. \le Abs\ z \Rightarrow \neg\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$
$\wedge\ c\ n = \mathbb{NC}\ 1$
$\Rightarrow S1S1Degree$
$((\lambda\ z \bullet\ z * \mathbb{RC}\ (Abs\ z\ ^{-1}))\ o\ Poly_C\ (c,\ n))$
$= \mathbb{NZ}\ n$

constant_term_0_root_thm

$\vdash \forall\ c\ n \bullet\ c\ 0 = \mathbb{NC}\ 0 \Rightarrow Poly_C\ (c,\ n)\ (\mathbb{NC}\ 0) = \mathbb{NC}\ 0$

monic_fta_thm

$\vdash \forall\ c\ n$
$\bullet\ c\ 0 = \mathbb{NC}\ 0 \vee \neg\ n = 0 \wedge c\ n = \mathbb{NC}\ 1$
$\Rightarrow (\exists\ z \bullet\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$

fta_thm

$\vdash \forall\ c\ n$
$\bullet\ c\ 0 = \mathbb{NC}\ 0 \vee \neg\ n = 0 \wedge \neg\ c\ n = \mathbb{NC}\ 0$
$\Rightarrow (\exists\ z \bullet\ Poly_C\ (c,\ n)\ z = \mathbb{NC}\ 0)$

INDEX

$*_C$ 4
$*_{RC}$ 7
$+_C$ 4
$-_C$ 6
$/_C$ 6
Abs_C 7
ClassS1Degree 10
constant_term_0_root_thm 21
de_moivre_thm 13
discrete_subgroup_ℝ_discrete_thm 19
ExpS1 9
exp_exp_s1_thm 16
exp_s1_2_π_thm 18
exp_s1_continuous_thm 15
exp_s1_covering_projection_lemma1 17
exp_s1_covering_projection_lemma2 17
exp_s1_covering_projection_lemma3 17
exp_s1_covering_projection_lemma4 17
exp_s1_covering_projection_lemma5 17
exp_s1_covering_projection_thm 17
exp_s1_homomorphism_thm1 16
exp_s1_homomorphism_thm 16
exp_s1_minus_thm 16
exp_s1_onto_thm1 17
exp_s1_onto_thm2 17
exp_s1_onto_thm3 17
exp_s1_onto_thm 16
exp_s1_path_fibration_thm 18
exp_s1_path_lifting_thm1 17
exp_s1_path_lifting_thm 17
exp_s1_period_thm1 16
exp_s1_period_thm 16
exp_s1_thm 18
exp_s1_unique_path_lifting_thm 17
exp_s1_ ∈ _s1_thm 16
Exp_C 7
fta_thm 21
IotaS1 10
iota_s1_continuous_thm 18
iota_s1_loop_thm 18
I_C 5
ker_exp_s1_discrete_thm 19
LoopS1Degree_consistent 18
LoopS1Degree 10
monic_fta_thm 21
one_ ∈ _s1_thm 19
open_s1_topology_thm 15
open_ℂ_const_continuous_thm 16
open_ℂ_hausdorff_thm 15
open_ℂ_id_continuous_thm 16
open_ℂ_topology_thm 15
$Open_C$ 9
$Open_{S1}$ 9
$PlusCoeffs_C$ 8
$PolyFunc_C$ 8
$Poly_C$ 8
pp_s1_continuous_thm 19
pp_s1_ ∈ _s1_thm 19
S1S1Degree 10
S1S1Loop 10
s1_s1_degree_const_thm 19
s1_s1_degree_eq_thm 19
s1_s1_degree_homotopy_invariant_lemma1 19
s1_s1_degree_homotopy_invariant_lemma2 19
s1_s1_degree_homotopy_invariant_thm 19
s1_s1_degree_ℂ_ℕ_exp_thm 19
s1_s1_loop_eq_thm 19
s1_s1_loop_loop_thm 18
s1_times_continuous_thm 19
s1_times_thm 16
S1 9
$Sigma_C$ 8
space_t_open_s1_thm 15
space_t_open_ℂ_thm 15
space_t_subspace_open_ℂ_thm 15
sqrt_ ≤ _1_thm 20
subspace_ℂ_topology_thm 15
$TimesCoeffs_C$ 9
times_2_π_lemma 18
translated_path_ℝ_path_thm 18
universe_open_ℂ_thm 15
$- 5
∈ _s1_lemma 15
$\hat{}_C$ 6
π_recip_clauses 18
ℂ_0_ ≤ _abs_thm 16
ℂ_abs_continuous_thm 16
ℂ_abs_eq_0_thm1 16
ℂ_abs_eq_0_thm 16
ℂ_abs_plus_thm 16
ℂ_abs_squared_lemma 15
ℂ_abs_times_thm 16
ℂ_abs_ℝ_abs_thm 19
ℂ_additive_group_thm 13
ℂ_additive_ops_thm 13
ℂ_additive_ℂ_multiplicative_homomorphism_def 13
ℂ_conj_continuous_thm 15
ℂ_conj_plus_homomorphism_thm 11
ℂ_conj_times_homomorphism_thm 11
ℂ_const_eval_thm 14
ℂ_const_times_eval_thm 14
ℂ_eq_thm 12
ℂ_eq_ℝ_ × _ℝ_thm 13
ℂ_exp_homomorphism_thm 13
ℂ_id_eval_thm 14
ℂ_linear_homomorphism_thm 13
ℂ_minus_clauses 12
ℂ_minus_continuous_thm 15
ℂ_multiplicative_group_thm 13
ℂ_multiplicative_ops_thm 13

$\mathbb{C}$_*plus_0_thm* 11
$\mathbb{C}$_*plus_assoc_thm1* 11
$\mathbb{C}$_*plus_assoc_thm* 11
$\mathbb{C}$_*plus_clauses* 12
$\mathbb{C}$_*plus_comm_thm* 11
$\mathbb{C}$_*plus_continuous_thm* 15
$\mathbb{C}$_*plus_eval_rw_thm* 14
$\mathbb{C}$_*plus_eval_thm* 14
$\mathbb{C}$_*plus_minus_thm* 11
$\mathbb{C}$_*plus_order_thm* 11
$\mathbb{C}$_*poly_0_thm* 14
$\mathbb{C}$_*poly_continuous_thm* 19
$\mathbb{C}$_*poly_degree_0_lemma1* 20
$\mathbb{C}$_*poly_degree_0_lemma2* 20
$\mathbb{C}$_*poly_degree_0_thm* 21
$\mathbb{C}$_*poly_degree_n_lemma1* 20
$\mathbb{C}$_*poly_degree_n_lemma2* 20
$\mathbb{C}$_*poly_degree_n_lemma3* 21
$\mathbb{C}$_*poly_degree_n_lemma4* 21
$\mathbb{C}$_*poly_degree_n_lemma5* 21
$\mathbb{C}$_*poly_degree_n_thm* 21
$\mathbb{C}$_*poly_eq_thm1* 14
$\mathbb{C}$_*poly_eq_thm2* 14
$\mathbb{C}$_*poly_eq_thm* 14
$\mathbb{C}$_*poly_func_continuous_thm* 15
$\mathbb{C}$_*poly_func_eq_poly_eval_thm* 15
$\mathbb{C}$_*poly_induction_thm* 15
$\mathbb{C}$_*poly_minus_thm* 14
$\mathbb{C}$_*poly_plus_thm* 14
$\mathbb{C}$_*poly_rec_thm* 14
$\mathbb{C}$_*poly_times_thm* 20
$\mathbb{C}$_*punctured_set_thm* 17
$\mathbb{C}$_*recip*_$\neg$_*eq_0_thm* 19
$\mathbb{C}$_*sigma_const_thm* 20
$\mathbb{C}$_*sigma_continuous_thm* 15
$\mathbb{C}$_*sigma_eq_thm* 20
$\mathbb{C}$_*sigma_rw_thm* 13
$\mathbb{C}$_*sigma_times_thm1* 20
$\mathbb{C}$_*sigma_times_thm* 20
$\mathbb{C}$_*times_0_thm* 12
$\mathbb{C}$_*times_1_thm* 11
$\mathbb{C}$_*times_assoc_thm1* 11
$\mathbb{C}$_*times_assoc_thm* 11
$\mathbb{C}$_*times_clauses* 12
$\mathbb{C}$_*times_comm_thm* 11
$\mathbb{C}$_*times_conj_thm* 11
$\mathbb{C}$_*times_continuous_thm* 15
$\mathbb{C}$_*times_eq_0_thm1* 12
$\mathbb{C}$_*times_eq_0_thm* 12
$\mathbb{C}$_*times_eq_1_thm* 11
$\mathbb{C}$_*times_eval_rw_thm* 15
$\mathbb{C}$_*times_eval_thm* 14
$\mathbb{C}$_*times_minus_thm* 12
$\mathbb{C}$_*times_order_thm* 12
$\mathbb{C}$_*times_plus_distrib_thm* 11
$\mathbb{C}$_*times_recip_thm1* 11
$\mathbb{C}$_*times_recip_thm* 11
$\mathbb{C}$_$\mathbb{N}$_*exp_0_thm* 12
$\mathbb{C}$_$\mathbb{N}$_*exp_1_thm* 12
$\mathbb{C}$_$\mathbb{N}$_*exp_clauses* 13
$\mathbb{C}$_$\mathbb{N}$_*exp_continuous_thm* 15
$\mathbb{C}$_$\mathbb{N}$_*exp_minus_thm* 13
$\mathbb{C}$_$\mathbb{N}$_*exp_plus_thm* 13
$\mathbb{C}$_$\mathbb{N}$_*exp_recip_thm* 13
$\mathbb{C}$_$\mathbb{N}$_*exp_rw_thm* 12
$\mathbb{C}$_$\mathbb{N}$_*exp_s1_s1_continuous_thm* 19
$\mathbb{C}$_$\mathbb{N}$_*exp_times_thm* 13
$\mathbb{C}$_$\mathbb{N}$_*exp*_ $\in$ _*s1_thm* 19
$\mathbb{C}$_$\mathbb{N}$_*exp*_$\neg$_*eq_0_thm* 13
$\mathbb{C}_*$ 7
$\mathbb{C}_+$ 7
$\mathbb{N}\iota$ 6
$\mathbb{N}\mathbb{C}$_*one_one_thm* 11
$\mathbb{N}\mathbb{C}$_*plus_homomorphism_thm* 11
$\mathbb{N}\mathbb{C}$_*times_homomorphism_thm* 11
$\mathbb{N}\mathbb{C}$ 6
$\mathbb{R}\iota$ 5
$\mathbb{R}$_0_ $\leq$ _*square_plus_square_thm* 16
$\mathbb{R}$_2*d_cauchy_schwarz_thm* 16
$\mathbb{R}$_2*d_triangle_inequality_thm* 16
$\mathbb{R}$_*recip*_$\mathbb{C}$_*abs_continuous_thm* 19
$\mathbb{R}$_*square_eq_0_thm* 16
$\mathbb{R}$_*square_plus_square_eq_0_thm* 11
$\mathbb{R}$_*times_mono*_ $\Leftrightarrow$ _*thm1* 18
$\mathbb{R}\mathbb{C}$_*plus_homomorphism_thm* 11
$\mathbb{R}\mathbb{C}$_*thm* 19
$\mathbb{R}\mathbb{C}$_*times_homomorphism_thm* 11
$\mathbb{R}\mathbb{C}$ 5
$\mathbb{Z}$_ $\leq$ _*cases_thm1* 18
$\mathbb{Z}\mathbb{R}$_0_*less_thm* 18
$\mathbb{Z}\mathbb{R}$_*less_thm* 18
$\mathbb{Z}\mathbb{R}$_*minus_homomorphism_thm* 18
$\mathbb{Z}\mathbb{R}$_*one_one_thm* 19
$\sim_C$ 4
$^{-1C}$ 5